SOLUTION OF THE DIRICHLET PROBLEM WITH A VARIATIONAL METHOD

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1. Dirichlet integral

Let \( f \in C(\partial \Omega) \) with \( \Omega \) open and bounded. Let \( H = \{ u \in C^1(\bar{\Omega}) : u = f \text{ on } \partial \Omega \} \) and

\[
D(u) = \int_{\Omega} |Du(x)|^2 \, dx.
\]

The objective is to prove that with minimizers of \( D(u) \) over \( H \) one can solve the Dirichlet problem in \( \Omega \).

We assume that \( f \) satisfies the following property: there exists \( v \in C^1(\bar{\Omega}) \) such that \( v = f \) on \( \partial \Omega \). This is not a restriction to solve the Dirichlet problem with this approach because if \( f \in C(\partial \Omega) \) then by the Weierstrass approximation theorem there exist polynomials \( f_k \) in \( \mathbb{R}^n \) such that \( f_k|_{\partial \Omega} \) converge uniformly to \( f \) on \( \partial \Omega \). If we can solve the Dirichlet problem with data \( u_k = f_k \) on \( \partial \Omega \), then by the maximum principle \( u_k \rightarrow u \) uniformly in \( \Omega \) for some \( u \) and therefore \( u \) is harmonic in \( \Omega \) and has boundary values \( f \).

If \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) satisfies \( \Delta u = 0 \) in \( \Omega \) and \( u = f \) on \( \partial \Omega \), then

\[
D(u) \leq D(v), \quad \forall v \in H.
\]

That is, the Dirichlet integral is minimized by the solution of the Dirichlet problem. This follows writing \( g = v - u \) with \( v \in H \)

\[
D(v) = D(g + u) = D(g) + 2D(g, u) + D(u)
\]

\[
= D(g) + D(u) + 2 \int_{\partial \Omega} g D_v u \, d\sigma - 2 \int_{\Omega} g \Delta u \, dx
\]

\[
= D(g) + D(u) \geq D(u)
\]

from the first Green formula. There are continuous functions \( f \in C(\partial \Omega) \) such that the solution \( u \) of the Dirichlet problem with data \( f \) satisfies \( D(u) = +\infty \). An
example is the function \( u(r, \theta) = \sum_{k=1}^{\infty} \frac{r^n \cos(n! \theta)}{n^2} \) that is harmonic in the unit disk, it has boundary values \( f(\theta) = \sum_{k=1}^{\infty} \frac{\cos(n! \theta)}{n^2} \), and \( D(u) = +\infty \).

We have that \( D(\lambda u + v) = \lambda^2 D(u) + 2\lambda D(u, v) + D(v) \geq 0 \) for all \( \lambda \), and so \( D(u)D(v) - D(u, v) \geq 0 \). Therefore \( \|u\|_D = D(u)^{1/2} \) defines a quasi norm in \( H \), that is, \( \| \cdot \|_D \) satisfies the triangle inequality and \( \| \lambda u \|_D = |\lambda| \|u\|_D \).

2. Poincaré inequality

Let \( H = \{ u \in C^1(\bar{\Omega}) : u = f \text{ on } \partial \Omega \} \) and \( H_0 = \{ u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial \Omega \} \).

**Lemma 1.** There exists a constant \( C > 0 \), depending only on the domain \( \Omega \), such that

\[
\int_{\Omega} w(x)^2 \, dx \leq C \int_{\Omega} |Dw(x)|^2 \, dx,
\]

for all \( w \in H_0 \).

**Proof.** Since \( \Omega \) is bounded, \( \Omega \subset Q = [-a, a] \times [-a, a] \). Let \( \bar{w}(x, y) = w(x, y) \) for \( (x, y) \in \Omega \) and \( \bar{w}(x, y) = 0 \) for \( (x, y) \in Q \setminus \Omega \). We assume that the intersection of \( \Omega \) with each vertical line is finite union of open intervals. Then the function \( \bar{w}(x, \cdot) \) is Lipschitz and therefore absolutely continuous. So we can write

\[
\bar{w}(x, y) = \int_{-a}^{y} \bar{w}_y(x, \xi) \, d\xi.
\]

Then squaring and using Cauchy-Schwartz we get

\[
\bar{w}(x, y)^2 \leq (y + a) \int_{-a}^{y} \bar{w}_y(x, \xi)^2 \, d\xi \leq 2a \int_{-a}^{a} \bar{w}_y(x, \xi)^2 \, d\xi,
\]

and integrating this inequality in \( x \) yields

\[
\int_{-a}^{a} \bar{w}(x, y)^2 \, dx \leq 2a \int_{-a}^{a} \int_{-a}^{a} \bar{w}_y(x, \xi)^2 \, d\xi \, dx.
\]

Now integrating in \( y \) yields

\[
\int_{-a}^{a} \int_{-a}^{a} \bar{w}(x, y)^2 \, dx \, dy \leq 4a^2 \int_{-a}^{a} \int_{-a}^{a} \bar{w}_y(x, \xi)^2 \, d\xi \, dx.
\]

From the definition of \( \bar{w} \) and since \( \bar{w}_y(x, y) = 0 \) when \( (x, y) \in Q \setminus \Omega \), the lemma then follows with \( C = 4a^2 \). \( \square \)
Remark 2. It is clear that if $\Omega \subset [a,b] \times [c,d]$, then the estimate holds with $C = (b-a)(d-c)$. In higher dimensions, one obtains in the same way that if $\Omega \subset \mathbb{R}$ with $\mathbb{R}$ an $n$-dimensional interval, then the lemma holds with $C = |\mathbb{R}|$.

3. Solution to the Dirichlet problem

By our assumption the set $H$, $\emptyset$, and therefore $\inf_H D(u) = L < \infty$. So there exists a sequence (possibly not unique) $v_k \in H$ such that $D(v_k) \to L$ as $k \to \infty$. We call $v_k$ a minimizing sequence, and the objective is to construct with $v_k$ a harmonic function $\phi$ in $\Omega$ such that $\phi = f$ on $\partial \Omega$.

Lemma 3. For each $w \in H_0$, we have
\[
\lim_{k \to \infty} D(v_k, w) = 0,
\]
where $v_k$ is the minimizing sequence. Moreover, if $D(w_k) \leq M$ with $w_k \in H_0$, then $D(v_k, w_k) \to 0$ as $k \to \infty$.

Proof. We have $v_k + \epsilon w \in H$ for all $\epsilon$ and so
\[
L \leq D(v_k + \epsilon w) = D(v_k) + 2\epsilon D(v_k, w) + \epsilon^2 D(w),
\]
and the minimum of the right hand side is attained when $\epsilon = -\frac{D(v_k, w)}{D(w)}$ which yields
\[
L \leq D(v_k) - \frac{D^2(v_k, w)}{D(w)}.
\]
Therefore
\[
|D(v_k, w)| \leq (D(v_k) - L)^{1/2} D(w)^{1/2},
\]
and the lemma follows. \hfill \Box

Lemma 4. The minimizing sequence $v_k$ is a Cauchy sequence in the norm $\|\cdot\|_{L^2(\Omega)} + \|\cdot\|_D$.

Proof. Let $w = v_k - v_m$. We have $D(v_k) = D(v_m + w) = D(v_m) + 2D(v_m, w) + D(w)$. So
\[
|D(w)| \leq |D(v_k) - D(v_m)| + 2|D(v_m, w)|
\]
and therefore from the previous lemma, $|w|_{L^2(\Omega)} \to 0$ as $k, m \to \infty$. From the Poincaré inequality, $|w|_{L^2(\Omega)} \leq C|w|_D$ and the lemma follows. \hfill \Box

Given $x \in \Omega$ and $B_{\rho}(x) \subset \Omega$ we let
\[
\phi_k(x, \rho) = \int_{B_{\rho}(x)} v_k(z) \, dz.
\]
Lemma 5. Let $K$ be closed $K \subset \Omega$, and fix $\rho < \text{dist}(K, \Omega^c)$. Then $\phi_k(x, \rho)$ are continuous and converge uniformly for $x \in K$ to a function $\phi(x, \rho)$ for each $\rho$.

Proof. The functions $\phi_k(x, \rho)$ are clearly continuous. We write

$$|\phi_k(x, \rho) - \phi_m(x, \rho)| = \left| \int_{B_{\rho}(x)} (v_k(z) - v_m(z)) \, dz \right| \leq \left( \int_{B_{\rho}(x)} |v_k(z) - v_m(z)|^2 \, dz \right)^{1/2} \leq \frac{1}{|B_{\rho}(x)|^{1/2}} \|v_k - v_m\|_{L^2(\Omega)},$$

so the sequence $\phi_k(x, \rho)$ is uniformly Cauchy and the lemma follows. \hfill \Box

Lemma 6. The function $\phi(x, \rho)$ is independent of $\rho$, i.e., $\phi(x, \rho) = \phi(x)$.

Proof. Fix $x \in \Omega$ and let $\rho_1 < \rho_2$ with $B_{\rho_1}(x) \subset \Omega$. Suppose we are in dimension two and let

$$w(x) = \begin{cases} \frac{1}{2\pi} \left( \log \frac{\rho_1}{\rho_2} + \frac{1}{2} |x - z|^2 \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) \right), & |x - z| \leq \rho_1 \\ \frac{1}{2\pi} \left( \log \frac{|x - z|}{\rho_2} + \frac{1}{2} \left( 1 - \frac{|x - z|^2}{\rho_2^2} \right) \right), & \rho_1 < |x - z| \leq \rho_2 \\ 0 & \rho_2 < |x - z|, \end{cases}$$

We have

$$Dw(x) = \begin{cases} \frac{1}{2\pi} (x - z) \left( \frac{1}{\rho_2^2} - \frac{1}{\rho_1^2} \right), & |x - z| \leq \rho_1 \\ \frac{1}{2\pi} (x - z) \left( \frac{1}{\rho_2^2} - \frac{1}{|x - z|^2} \right), & \rho_1 < |x - z| \leq \rho_2 \\ 0 & \rho_2 < |x - z|, \end{cases}$$

We have that $w \in H_0$ and

$$\Delta w(x) = \begin{cases} \frac{1}{\pi \rho_1^2} - \frac{1}{\pi \rho_2^2}, & |x - z| < \rho_1 \\ -\frac{1}{\pi \rho_2^2}, & \rho_1 < |x - z| < \rho_2 \\ 0 & \rho_2 < |x - z|. \end{cases}$$

To apply the first Green formula we remove the places where the Laplacian of $w$ is discontinuous. We set

$$\Omega_\epsilon = \Omega \setminus \{ z : \rho_1 - \epsilon < |x - z| < \rho_1 + \epsilon \} \cup \{ z : \rho_2 - \epsilon < |x - z| < \rho_2 + \epsilon \}.$$
By the first Green formula applied in $\Omega_\epsilon$ we have
\[ \int_{\Omega_\epsilon} Dv_k \cdot Dw \, dx + \int_{\Omega_\epsilon} v_k \Delta w \, dx = \int_{\partial \Omega_\epsilon} v_k(z) \partial_\eta w(x) \, d\sigma(z). \]

From the form of the gradient of $w$, the right hand side of the last identity tends to zero as $\epsilon \to 0$, and so we obtain
\[ 0 = D(v_k, w) + \int_{\Omega} v_k \Delta w \, dx = D(v_k, w) + \int_{B_{\rho_1}(x)} v_k(z) \, dz - \int_{B_{\rho_2}(x)} v_k(z) \, dz, \]
and letting $k \to \infty$ the lemma follows. \hfill \Box

**Theorem 7.** The function $\phi$ is harmonic in $\Omega$ and $\phi = f$ on $\partial \Omega$.

**Proof.** We claim that if $B_a(x_0), B_b(x_0) \subset \Omega$, then
\[ \int_{B_a(x_0)} \phi_k(x, a) \, dx = \int_{B_b(x_0)} \phi_k(x, b) \, dx. \]

In fact, we have
\[ \phi_k(x, a) = \int_{B_a(x)} v_k(z) \, dz = \int_{B_a(0)} v_k(z + x) \, dz, \]
and integrating in $x$
\[ \int_{B_a(x_0)} \phi_k(x, a) \, dx = \int_{B_a(x_0)} \int_{B_a(0)} v_k(u) \, du \, dz = \int_{B_a(0)} \int_{B_a(x_0)} v_k(u + z) \, du \, dz \]
\[ = \int_{B_a(0)} \int_{B_b(x_0 + z)} v_k(u) \, du \, dz = \int_{B_a(0)} \phi_k(x_0 + z, b) \, dz \]
\[ = \int_{B_b(x_0)} \phi_k(z, b) \, dz, \]
which proves the claim. Letting $k \to \infty$ we obtain
\[ \int_{B_a(x_0)} \phi(x) \, dx = \int_{B_b(x_0)} \phi(x) \, dx, \]
and since $\phi$ is continuous, letting $b \to 0$ yields
\[ \int_{B_a(x_0)} \phi(x) \, dx = \phi(x_0), \]
that is, $\phi$ satisfies the mean value property in $\Omega$ and the first part of the theorem is then proved.

We shall prove in dimension two that
\[ \lim_{x \to x_0, x \in \Omega} \phi(x) = f(x_0), \quad x_0 \in \partial \Omega, \]
under the following assumption on \( \Omega \): there exists \( R > 0 \) such that \( \{ z : |z - x| = \rho \} \cap \partial \Omega \neq \emptyset \) for all \( x \in \partial \Omega \) and for all \( \rho \leq R \). Let \( x_0 \in \partial \Omega \) and consider the ball \( B_\rho(x_0) \), and let \( x \in B_{\rho/2}(x_0) \cap \Omega \). Let \( \sigma = \text{dist}(x, \partial \Omega) \). There exists \( x_1 \in \partial \Omega \) such that \( |x_1 - x| = \sigma \). We have \( \sigma \leq h/2, |x_0 - x_1| < h \), and \( B_{\sigma/2}(x) \subset B_{3\sigma/2}(x_1) \cap \Omega \). We write

\[
|f(x_0) - \phi(x)| \leq |f(x_0) - f(x_1)| + |f(x_1) - \phi(x)| + \left| v_k(x) - \int_{B_{\sigma/2}(x)} v_k(z) \, dz \right| + \left| \int_{B_{\sigma/2}(x)} v_k(z) \, dz - \phi(x) \right|
\]

\[
= I + II + III + IV.
\]

Since \( f \) is continuous, \( I \) is small for \( h \) small. \( II \) is small since \( v_k \in C(\bar{\Omega}) \) and \( v_k = f \) on \( \partial \Omega \). \( III \) is also small for \( h \) small since \( \sigma \leq h/2 \).

We estimate \( IV \). Consider the circle \( C_\rho(x_1) \) for \( \rho \leq 3\sigma/2 \). If \( 3\sigma/2 < R \), where \( R \) appears in the condition on \( \Omega \), then \( C_\rho(x_1) \) intersects \( \partial \Omega \) at some point \( x_\rho \). Let \( w = v_k - v_m \). We have \( w \in H_0(\Omega) \) and we extend \( w \) to be zero outside \( \bar{\Omega} \) and we still call this extension \( w \). Let \( h(\theta) = w(x_1 + \rho(\cos \theta, \sin \theta)) \), \( x' \in C_\rho(x_1) \), and \( x_1 + \rho(\cos \theta_0, \sin \theta_0) = x_\rho, x_1 + \rho(\cos \theta_1, \sin \theta_1) = x' \). We have

\[
w(x') = \int_{\theta_0}^{\theta_1} h'(\theta) \, d\theta = \int_{\theta_0}^{\theta_1} Dw(x_1 + \rho(\cos \theta, \sin \theta)) \cdot (-\sin \theta, \cos \theta) \rho \, d\theta.
\]

Then

\[
|w(x')| \leq \int_{\theta_0}^{\theta_1} |Dw(x_1 + \rho(\cos \theta, \sin \theta))| \rho \, d\theta
\]

\[
\leq \rho|\theta_1 - \theta_0|^{1/2} \left( \int_{\theta_0}^{\theta_1} |Dw(x_1 + \rho(\cos \theta, \sin \theta))|^2 \, d\theta \right)^{1/2}
\]

\[
\leq \rho(2\pi)^{1/2} \left( \int_0^{2\pi} |Dw(x_1 + \rho(\cos \theta, \sin \theta))|^2 \, d\theta \right)^{1/2},
\]

that is,

\[
|w(x_1 + \rho(\cos \theta_1, \sin \theta_1))|^2 \leq 2\pi \rho^2 \int_0^{2\pi} |Dw(x_1 + \rho(\cos \theta, \sin \theta))|^2 \, d\theta.
\]

Integrating this inequality for \( 0 \leq \theta_1 \leq 2\pi \) yields

\[
\int_0^{2\pi} |w(x_1 + \rho(\cos \theta_1, \sin \theta_1))|^2 \, d\theta_1 \leq 4\pi^2 \rho^2 \int_0^{2\pi} |Dw(x_1 + \rho(\cos \theta, \sin \theta))|^2 \, d\theta,
\]

and now integrating for \( 0 \leq \rho \leq 3\sigma/2 \) we obtain

\[
\int_{B_{3\sigma/2}(x_1)} w(z)^2 \, dz \leq 9\pi^2 \sigma^2 \int_{B_{3\sigma/2}(x_1)} |Dw(z)|^2 \, dz.
\]
Therefore
\[
\left| \int_{B_{\sigma/2}(x)} (v_k(z) - v_m(z)) \, dz \right| \leq \left( \int_{B_{\sigma/2}(x)} |v_k(z) - v_m(z)|^2 \, dz \right)^{1/2} \\
\leq 3 \left( \int_{B_{3\sigma/2}(x_1)} |w(z)|^2 \, dz \right)^{1/2} \\
\leq 2 \sqrt{\pi} \left( \int_{B_{3\sigma/2}(x_1)} |Dw(z)|^2 \, dz \right)^{1/2} \\
\leq 2 \sqrt{\pi} \left( \int_{\Omega} |D(v_k - v_m)(z)|^2 \, dz \right)^{1/2}
\]
which tends to zero as \( k, m \to \infty \). Letting \( m \to \infty \) we obtain
\[
\left| \int_{B_{\sigma/2}(x)} v_k(z) \, dz - \phi(x) \right| < \epsilon
\]
for all \( k \) sufficiently large and so \( IV \) is also small.

\[\square\]