1. The refractor problem $\kappa < 1$

Let $n_1$ and $n_2$ be the indexes of refraction of two homogeneous and isotropic media I and II, respectively. Suppose that from a point $O$ inside medium I light emanates with intensity $f(x)$ for $x \in \Omega$. We want to construct a refracting surface $\mathcal{R}$ parameterized as $\mathcal{R} = \{\rho(x) : x \in \Omega\}$, separating media I and II, and such that all rays refracted by $\mathcal{R}$ into medium II have directions in $\Omega^*$ and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$.

We first introduce the notions of refractor mapping and measure, and weak solution.

Let $\Omega, \Omega^*$ be two domains on $S^{n-1}$, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\Omega)$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure $\mu$ on $\Omega^*$. Throughout the paper, we assume that $|\partial \Omega| = 0$ and the physical constraint

$$\inf_{x \in \Omega, m \in \Omega^*} x \cdot m \geq \kappa. \tag{1.1}$$

We further suppose that the total energy conservation

$$\int_{\Omega} f(x) \, dx = \mu(\Omega^*) > 0, \tag{1.2}$$

and for any open set $G \subset \Omega$

$$\int_{G} f(x) \, dx > 0, \tag{1.3}$$

where $dx$ denotes the surface measure on $S^{n-1}$. 

1.1. **Refractor measure and weak solutions.** We begin with the notions of refractor and supporting semi-ellipsoid.

**Definition 1.1.** A surface $\mathcal{R}$ parameterized by $\rho(x)$ with $\rho \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega'}$ for the case $\kappa < 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m, b)$ with $m \in \overline{\Omega'}$ such that $\rho(x_0) = \frac{b}{1 - \kappa m \cdot x_0}$ and $\rho(x) \leq \frac{b}{1 - \kappa m \cdot x}$ for all $x \in \overline{\Omega}$. Such $E(m, b)$ is called a supporting semi-ellipsoid of $\mathcal{R}$ at the point $\rho(x_0)x_0$.

From the definition, any refractor is globally Lipschitz on $\overline{\Omega}$.

**Definition 1.2.** Given a refractor $\mathcal{R} = \{\rho(x) : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$

$$N_{\mathcal{R}}(x_0) = \{m \in \overline{\Omega'} : E(m, b) \text{ supports } \mathcal{R} \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}.$$  

Given $m_0 \in \overline{\Omega'}$, the tracing mapping of $\mathcal{R}$ is defined by

$$T_{\mathcal{R}}(m_0) = N_{\mathcal{R}}^{-1}(m_0) = \{x \in \overline{\Omega} : m_0 \in N_{\mathcal{R}}(x)\}.$$

The next lemmas discuss the refractor measure.

**Lemma 1.3.** $C = \{F \subset \overline{\Omega'} : T_{\mathcal{R}}(F) \text{ is Lebesgue measurable}\}$ is a $\sigma$-algebra containing all Borel sets in $\overline{\Omega'}$.

**Proof.** Obviously, $T_{\mathcal{R}}(\emptyset) = \emptyset$ and $T_{\mathcal{R}}(\overline{\Omega'}) = \overline{\Omega}$. Since $T_{\mathcal{R}}(\bigcup_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} T_{\mathcal{R}}(F_i)$, $C$ is closed under countable unions. Clearly for $F \subset \overline{\Omega'}$

$$T_{\mathcal{R}}(F^c) = \{x \in \overline{\Omega} : N_{\mathcal{R}}(x) \cap F^c \neq \emptyset\}$$

$$= \{x \in \overline{\Omega} : N_{\mathcal{R}}(x) \cap F = \emptyset\} \cup \{x \in \overline{\Omega} : N_{\mathcal{R}}(x) \cap F^c \neq \emptyset, N_{\mathcal{R}}(x) \cap F \neq \emptyset\}$$

(1.4) $$= [T_{\mathcal{R}}(F)]^c \cup [T_{\mathcal{R}}(F^c) \cap T_{\mathcal{R}}(F)].$$

If $x \in T_{\mathcal{R}}(F^c) \cap T_{\mathcal{R}}(F) \cap \Omega$, then $\mathcal{R}$ parameterized by $\rho$ has two distinct supporting semi-ellipsoids $E(m_1, b_1)$ and $E(m_2, b_2)$ at $\rho(x)x$. We show that $\rho(x)x$ is a singular point of $\mathcal{R}$. Otherwise, if $\mathcal{R}$ has the tangent hyperplane $\Pi$ at $\rho(x)x$, then $\Pi$ must
coincide both with the tangent hyperplane of \( E(m_1, b_1) \) and that of \( E(m_2, b_2) \) at \( \rho(x)x \). It follows from the Snell law that \( m_1 = m_2 \). Therefore, the area measure of \( T_R(F^c) \cap T_R(F) \) is 0. So \( C \) is closed under complements, and we have proved that \( C \) is a \( \sigma \)-algebra.

To prove that \( C \) contains all Borel subsets, it suffices to show that \( T_R(K) \) is compact if \( K \subset \overline{\Omega}^\ast \) is compact. Let \( x_i \in T_R(K) \) for \( i \geq 1 \). There exists \( m_i \in N_R(x_i) \cap K \). Let \( E(m_i, b_i) \) be the supporting semi-ellipsoid to \( R \) at \( \rho(x_i)x_i \). We have

\[
\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \quad \text{for } x \in \overline{\Omega},
\]

where the equality in (1.5) occurs at \( x = x_i \). Assume that \( a_1 \leq \rho(x) \leq a_2 \) on \( \overline{\Omega} \) for some constants \( a_2 \geq a_1 > 0 \). By (1.5) and (1.1), \( a_1(1 - \kappa) \leq b_i \leq a_2(1 - \kappa^2) \). Assume through subsequence that \( x_i \longrightarrow x_0, m_i \longrightarrow m_0 \in K, b_i \longrightarrow b_0 \), as \( i \longrightarrow \infty \). By taking limit in (1.5), one obtains that the semi-ellipsoid \( E(m_0, b_0) \) supports \( R \) at \( \rho(x_0)x_0 \) and \( x_0 \in T_R(m_0) \). This proves \( T_R(K) \) is compact. \( \square \)

**Lemma 1.4.** We have

(i) \( [T_R(F)]^c \subset T_R(F^c) \) for all \( F \subset \overline{\Omega}^\ast \), with equality except for a set of measure zero.

(ii) The set \( C = \{ F \subset \overline{\Omega}^\ast : T_R(F) \text{ is Lebesgue measurable} \} \) is a \( \sigma \)-algebra containing all Borel sets in \( \overline{\Omega}^\ast \).

**Lemma 1.5.** Let \( R_j = \{ \rho_j(x)x : x \in \overline{\Omega} \}, j \geq 1 \) be refractors from \( \overline{\Omega} \) to \( \overline{\Omega}^\ast \). Suppose that \( 0 < a_1 \leq \rho_j \leq a_2 \) and \( \rho_j \rightarrow \rho \) uniformly on \( \overline{\Omega} \). Then:

(i) \( R := \{ \rho(x)x : x \in \overline{\Omega} \} \) is a refractor from \( \overline{\Omega} \) to \( \overline{\Omega}^\ast \).

(ii) For any compact set \( K \subset \overline{\Omega}^\ast \)

\[
\limsup_{j \to \infty} T_R(K) \subset T_R(K).
\]

(iii) For any open set \( G \subset \overline{\Omega}^\ast \),

\[
T_R(G) \subset \liminf_{j \to \infty} T_{R_j}(G) \cup S,
\]

where \( S \) is the singular set of \( R \).
Proof. (i) Obviously \( \rho \in C(\Omega) \) and \( \rho > 0 \). Fix \( x_o \in \Omega \). Then there exist \( m_j \in \Omega^* \) and \( b_j > 0 \) such that \( E(m_j, b_j) \) supports \( R_j \) at \( \rho(x_o)x_o \) and thus

\[
\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \quad \text{and} \quad \rho_j(x) \leq \frac{b_j}{1 - \kappa m_j \cdot x}
\]

for all \( x \in \Omega \). Consequently

\[
\frac{b_j}{1 - \kappa m_j \cdot x_o} \leq a_2 \quad \text{and} \quad a_1 \leq \frac{b_j}{1 - \kappa m_j \cdot x}
\]

for all \( j \) and therefore

\[
a_1(1 - \kappa) \leq b_j \leq a_2
\]

for all \( j \). If need be by passing to a subsequence we obtain \( m_o \) and \( b_o \) such that \( m_j \to m_o \in \Omega^* \) and \( b_j \to b_o \). We claim \( E(m_o, b_o) \) supports \( R \) at \( \rho(x_o)x_o \). Indeed

\[
\rho(x_o) = \lim_{j} \rho_j(x_o) = \lim_{j} \frac{b_j}{1 - \kappa m_j \cdot x_o} = \frac{b_o}{1 - \kappa m_o \cdot x_o}
\]

and

\[
\rho(x) = \lim_{j} \rho_j(x) \leq \lim_{j} \frac{b_j}{1 - \kappa m_j \cdot x} = \frac{b_o}{1 - \kappa m_o \cdot x}
\]

for all \( x \in \Omega \). Thus \( R \) is a refractor.

(ii) Let \( x_o \in \limsup \mathcal{T}_{R_j}(K) \). Without loss of generality assume that \( x_o \in \mathcal{T}_{R_j}(K) \) for all \( j \geq 1 \). Then there exist \( m_j \in \mathcal{N}_{R_j}(x_o) \cap K \) and \( b_j \) such that

\[
\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \quad \text{and} \quad \rho_j(x) \leq \frac{b_j}{1 - \kappa m_j \cdot x}
\]

for all \( x \in \Omega \). As in the proof of (i) we may assume that \( m_j \to m_o \in K \) and \( b_j \to b_o \) and conclude that \( E(m_o, b_o) \) supports \( R \) at \( \rho(x_o)x_o \), proving that \( x_o \in \mathcal{T}_{R}(m_o) \). Hence \( x_o \in \mathcal{T}_{R}(K) \).

(iii) Let \( G \) be an open subset of \( \Omega^* \). By (ii) \( \limsup \mathcal{T}_{R_j}(G^c) \subset \mathcal{T}_{R}(G^c) \) as \( G^c \) is compact. Also

\[
(1.6) \quad \limsup_{j \to \infty} \mathcal{T}_{R_j}(G)^c \subset \limsup_{j \to \infty} (\mathcal{T}_{R_j}(G)^c \cup \mathcal{T}_{R_j}(G) \cap \mathcal{T}_{R_j}(G^c))
\]
and by Lemma 1.4 the right hand side of (1.6) is equal to \( \limsup_{j \to \infty} \mathcal{T}_R(G^c) \). By (ii) we will then have
\[
\limsup_{j \to \infty} [\mathcal{T}_R(G)]^c \subset \mathcal{T}_R(G^c) = [\mathcal{T}_R(G)]^c \cup [\mathcal{T}_R(G) \cap \mathcal{T}_R(G^c)]^c.
\]
Taking complements we obtain
\[
\{\limsup_{j \to \infty} [\mathcal{T}_R(G)]^c\}^c \supset [\mathcal{T}_R(G)] \cap [\mathcal{T}_R(G) \cap \mathcal{T}_R(G^c)]^c.
\]
Consequently
\[
\liminf_{j \to \infty} \mathcal{T}_R(G) \supset [\mathcal{T}_R(G)] \cap [\mathcal{T}_R(G) \cap \mathcal{T}_R(G^c)]^c
\]
and thus
\[
[[\mathcal{T}_R(G)] \cap [\mathcal{T}_R(G) \cap \mathcal{T}_R(G^c)]^c] \cup S \subset \liminf_{j \to \infty} \mathcal{T}_R(G) \cup S.
\]
But \( \mathcal{T}_R(G) \cap \mathcal{T}_R(G^c) \subset S \). Thus
\[
\mathcal{T}_R(G) \subset \mathcal{T}_R(G) \cup S \subset \liminf_{j \to \infty} \mathcal{T}_R(G) \cup S
\]
as required. \( \square \)

**Lemma 1.6.** Given a nonnegative \( f \in L^1(\Omega) \), the set function
\[
\mathcal{M}_{R,f}(F) = \int_{\mathcal{T}_R(F)} f \, dx
\]
is a finite Borel measure defined on \( C \) and is called the refractor measure associated with \( R \) and \( f \).

**Proof.** Let \( \{F_i\}_{i=1}^{\infty} \) be a sequence of pairwise disjoint sets in \( C \). Let \( H_1 = \mathcal{T}_R(F_1) \) and \( H_k = \mathcal{T}_R(F_k) \setminus \bigcup_{i=1}^{k-1} \mathcal{T}_R(F_i) \), for \( k \geq 2 \). Since \( H_i \cap H_j = \emptyset \) for \( i \neq j \) and \( \bigcup_{k=1}^{\infty} H_k = \bigcup_{k=1}^{\infty} \mathcal{T}_R(F_k) \), it is easy to get
\[
\mathcal{M}_{R,f}(\bigcup_{k=1}^{\infty} F_k) = \int_{\bigcup_{k=1}^{\infty} H_k} f \, dx = \sum_{k=1}^{\infty} \int_{H_k} f \, dx.
\]
Observe that \( \mathcal{T}_R(F_k) \setminus H_k = \mathcal{T}_R(F_k) \cap (\bigcup_{i=1}^{k-1} \mathcal{T}_R(F_i)) \) is a subset of the singular set of \( R \) and has area measure 0 for \( k \geq 2 \). Therefore, \( \int_{H_k} f \, dx = \mathcal{M}_{R,f}(F_k) \) and the \( \sigma \)-additivity of \( \mathcal{M}_{R,f} \) follows. \( \square \)
The notion of weak solutions is introduced through the conservation of energy.

**Definition 1.7.** A refractor $R$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity $\mu$ on $\overline{\Omega}^*$ if for any Borel set $F \subset \overline{\Omega}^*$

\begin{equation}
\mathcal{M}_{R,f}(F) = \int_{T_R(F)} f(x) \, dx = \mu(F).
\end{equation}

**Remark 1.8** (Invariance by dilations). Suppose that $R$ is a refractor weak solution in the sense of Definition 1.7 with intensities $f, \mu$ and defined by $\rho(x)$ for $x \in \overline{\Omega}$. Then for each $\alpha > 0$, the refractor $\alpha R$ defined by $\alpha \rho(x)$ for $x \in \overline{\Omega}$ is a weak solution in the sense of Definition 1.7 with the same intensities. In fact, $E(m, b)$ is a supporting ellipsoid to $R$ at the point $y$ if and only if $E(\alpha m, \alpha b)$ is a supporting ellipsoid to $\alpha R$ at the point $y$. This means that $T_R(m) = T_{\alpha R}(m)$ for each $m \in \overline{\Omega}^*$.

2. Existence of solutions

2.1. Solution in the discrete case.

**Theorem 2.1.** Let $f \in L^1(\Omega)$ with $\inf_{\Omega} f > 0$, $g_1, \ldots, g_N$ positive numbers, $m_1, \ldots, m_N \in \Omega$ distinct points, $N \geq 2$, with $x \cdot m_j \geq \kappa$ for all $x \in \Omega$ and $1 \leq j \leq N$. Let $\mu = \sum_{j=1}^{N} g_j \delta_{m_j}$ and assume the conservation of energy condition

\begin{equation}
\int_{\Omega} f(x) \, dx = \mu(\Omega^*).
\end{equation}

Then there exists a refractor $R$ such that

(a) $\overline{\Omega} = \bigcup_{j=1}^{N} T_R(m_j)$,

(b) $\int_{T_R(m_j)} f(x) \, dx = g_j$ for $1 \leq j \leq N$.

To prove the theorem, we prove first a sequence of lemmas.

**Lemma 2.2.** Let

\begin{equation}
W = \left\{ b = (b_2, \ldots, b_N) : b_j > 0, \mathcal{M}_{R_b,f}(m_j) = \int_{T_{R_b}(m_j)} f(x) \, dx \leq g_j, j = 2, \ldots, N \right\},
\end{equation}

...
If $b$ from (2.1). Hence a singular point. And therefore,

$$
|T_m \sum R \text{ supporting ellipsoid to long as } b \text{ supports } R \text{ and so } b \in \mathbb{M} \text{.}
$$

So $T$ set $\rho(2.3)$

(2.3) \quad \rho(x) = R(b)(x) = \min_{1 \leq j \leq N} \frac{b_j}{1 - \kappa \cdot m_j}.

Then, with the assumptions of Theorem 2.1 we have

(a) $W \neq \emptyset$

(b) if $b = (1, b_2, \cdots, b_N) \in W$, then $b_j > \frac{1}{1 + \kappa}$ for $j = 2, \cdots, N$.

Proof. (a) If for some $j \neq 1$, the semi-ellipsoid $E(m_j, b)$ supports $R(b)$ at some $x \in \Omega$, then $\rho(z) \leq \frac{b}{1 - \kappa \cdot m_j}$ for all $z \in \Omega$, and $\rho(x) = \frac{b}{1 - \kappa \cdot m_j}$. Since $x \cdot m_j \geq \kappa$, we have

$$
\frac{b}{1 - \kappa^2} \leq \frac{b}{1 - \kappa \cdot x \cdot m_j} = \rho(x) \leq \frac{1}{1 - \kappa \cdot x \cdot m_j} \leq \frac{1}{1 - \kappa},
$$

and so $b \leq 1 + \kappa$. Therefore, if $b_i > 1 + \kappa$ for $2 \leq i \leq N$, then $E(m_i, b_i)$ cannot be a supporting ellipsoid to $R(b)$ at any $x \in \Omega$. On the other hand, if $x \in T_R(m_j)$, then $m_j \in N_R(x)$ and if $x$ is not a singular point of $R(b)$ there is a unique ellipsoid $E(m_j, b)$ supporting $R$ at $x$. But from the definition of $R$ there is an ellipsoid $E(m_k, b_k)$ that supports $R$ at $x$, and so $E(m_k, b_k) = E(m_j, b)$, i.e., $k = j$ and $b = b_j$. Consequently the set $T_R(m_i)$ is contained in the set of singular points and therefore has measure zero. So $M_{R(b,f)}(m_j) = 0 < g_j$ for $j = 2, \cdots, N$ and so any point $b = (1, b_2, \cdots, b_N) \in W$ as long as $b_i > 1 + \kappa$ for $i = 2, \cdots, N$.

(b) First notice that if $E(m_j, b_j)$ and $E(m_k, b_k)$ support $R(b)$ at $x_0$, then $x_0$ is a singular point. And therefore, $|T_R(m_j) \cap T_R(m_k)| = 0$ for $k \neq j$.

Claim 1. If $b \in W$, then $g_1 \leq M_{R(b,f)}(m_1)$.

Indeed,

$$
\sum_{i=1}^{N} M_{R(b,f)}(m_i) = \sum_{i=1}^{N} \int_{T_R(b)(m_i)} f(x) \, dx = \int_{\bigcup_{i=1}^{N} T_R(b)(m_i)} f(x) \, dx = \int_{\Omega} f(x) \, dx = \mu(\Omega^*) = \sum_{i=1}^{N} g_i,
$$

from (2.1). Hence

$$
g_1 - M_{R(b,f)}(m_1) + \sum_{i=2}^{N} (g_i - M_{R(b,f)}(m_i)) = 0.
$$

If $b \in W$, then $g_i - M_{R(b,f)}(m_i) \geq 0$ for $i = 2, \cdots, N$, and Claim 1 follows.
Claim 2. For each \( b \in W, \mathcal{T}_{R(b)}(m_1) \cap \left( \bigcup_{i=2}^{N} \mathcal{T}_{R(b)}(m_i) \right)^c \neq \emptyset. \)

Otherwise, \( \mathcal{T}_{R(b)}(m_1) \subset \bigcup_{i=2}^{N} \mathcal{T}_{R(b)}(m_i) \) which means that each point in \( \mathcal{T}_{R(b)}(m_1) \) is singular, and therefore \(|\mathcal{T}_{R(b)}(m_1)| = 0\). This contradicts Claim 1, since \( g_1 > 0 \).

Therefore, if \( b \in W \), then we can pick \( x_0 \in \mathcal{T}_{R(b)}(m_1) \cap \left( \bigcup_{i=2}^{N} \mathcal{T}_{R(b)}(m_i) \right)^c \) and so

\[
\rho(x_0) = \frac{1}{1 - \kappa x_0 \cdot m_1} < \frac{b_i}{1 - \kappa x_0 \cdot m_i}, \quad i = 2, \ldots, N
\]

so

\[
b_i > \frac{1 - \kappa x_0 \cdot m_i}{1 - \kappa x_0 \cdot m_1} \geq \frac{1 - \kappa x_0 \cdot m_i}{1 - \kappa^2} \geq \frac{1 - \kappa}{1 - \kappa^2} = \frac{1}{1 + \kappa}.
\]

\( \square \)

Lemma 2.3. If \( b_j = (b_j^1, \ldots, b_j^N) \rightarrow b_0 = (b_0^1, \ldots, b_0^N) \) as \( j \rightarrow \infty \), then \( \rho_j = \mathcal{R}(b_j) \rightarrow \rho_0 = \mathcal{R}(b_0) \) uniformly in \( \Omega \) as \( j \rightarrow \infty \).

Proof. Given \( y \in \bar{\Omega} \), there exists \( 1 \leq \ell \leq N \) such that \( \rho_0(y) = \frac{b_0^\ell}{1 - \kappa y \cdot m_\ell} \). Hence

\[
\rho_j(y) - \rho_0(y) \leq \frac{b_j^\ell}{1 - \kappa y \cdot m_\ell} - \frac{b_0^\ell}{1 - \kappa y \cdot m_\ell} \leq \frac{|b_j^\ell - b_0^\ell|}{1 - \kappa y \cdot m_\ell} \leq \frac{|b_j^\ell - b_0^\ell|}{1 - \kappa} \rightarrow 0,
\]
as \( j \rightarrow \infty \). \( \square \)

Lemma 2.4. Let \( \delta > 0 \) and the region \( R_\delta = \{(1, b_2, \ldots, b_N) : b_j \geq \delta, 2 \leq j \leq N\} \). The functions \( G_{R(b)}(m_i) \) are continuous for \( b \in R_\delta \) for \( i = 1, 2, \ldots, N \).

Proof. Let \( b_j = (1, b_j^2, \ldots, b_j^N) \in R_\delta \) with \( b_j \rightarrow b_0 \) as \( j \rightarrow \infty \). By Lemma 2.3, \( \rho_j \rightarrow \rho_0 \) uniformly in \( \bar{\Omega} \). Given \( x \in \Omega \), we have \( \rho_j(x) = \frac{b_j^\ell}{1 - \kappa x \cdot m_\ell} \) for some \( 1 \leq \ell \leq N \) and so \( \rho_j(x) \geq \frac{\min[1, \delta]}{1 + \kappa} \). On the other hand, \( \rho_j(x) = \min_{1 \leq \ell \leq N} \frac{b_j^\ell}{1 - \kappa x \cdot m_\ell} \leq \frac{1}{1 - \kappa x \cdot m_1} \leq \frac{1}{1 - \kappa} \). Therefore

\[
\frac{\min[1, \delta]}{1 + \kappa} \leq \rho_j(x) \leq \frac{1}{1 - \kappa}
\]

for all \( x \in \bar{\Omega} \) and for all \( j \).

Let us fix \( 1 \leq i \leq N \). Let \( G \subset \bar{\Omega} \) be a neighborhood of \( m_i \) such that \( m_\ell \notin G \) for \( \ell \neq i \). If \( x_0 \in \mathcal{T}_{R(b_j)}(G) \) and \( x_0 \) is not a singular point, then there exists a unique

\[
\rho(x_0) = \frac{1}{1 - \kappa x_0 \cdot m_1} < \frac{b_i}{1 - \kappa x_0 \cdot m_i}, \quad i = 2, \ldots, N
\]
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\[ m \in G \text{ and } b > 0 \text{ such that } \rho_j(x_0) = \frac{b}{1 - \kappa x_0 \cdot m} \text{ and } \rho_j(z) \leq \frac{b}{1 - \kappa z \cdot m} \text{ for all } x \in \Omega. \]

From the definition of \( R(b_j) \) and since \( x_0 \) is not singular, \( m = m_\ell \). Since \( m \in G \), we get \( m = m_i \). Therefore

\[ T_{R(b_j)}(G) \subset T_{R(b_j)}(m_i) \cup S, \]

where \( S \) is the set of singular points. By Lemma 1.5

\[ T_{R(b_0)}(G) \subset \liminf_{j \to \infty} T_{R(b_j)}(G) \cup S, \]

and we therefore obtain

\[ T_{R(b_0)}(G) \subset \liminf_{j \to \infty} T_{R(b_j)}(m_i) \cup S. \]

Thus

\[
\int_{T_{R(b_j)}(m_i)} f(x) \, dx \leq \int_{T_{R(b_0)}(G)} f(x) \, dx \leq \int_{\liminf_{j \to \infty} T_{R(b_j)}(m_i)} f(x) \, dx
\]

\[
\leq \liminf_{j \to \infty} \int_{T_{R(b_j)}(m_i)} f(x) \, dx \quad \text{by Fatou.}
\]

We next prove that

\[
\limsup_{j \to \infty} \int_{T_{R(b_j)}(m_i)} f(x) \, dx \leq \int_{T_{R(b_0)}(m_i)} f(x) \, dx.
\]

By Lemma 1.5

\[
\limsup_{j \to \infty} T_{R(b_j)}(K) \subset T_{R(b_0)}(K)
\]

for each \( K \) compact. Hence

\[
\int \limsup_{j \to \infty} T_{R(b_j)}(m_i) f(x) \, dx \leq \int T_{R(b_0)}(m_i) f(x) \, dx.
\]

By reverse Fatou we have

\[
\limsup_{j \to \infty} \int_{T_{R(b_j)}(m_i)} f(x) \, dx \leq \int \limsup_{j \to \infty} T_{R(b_j)}(m_i) f(x) \, dx
\]

and therefore the lemma is proved. \( \square \)
Proof of Theorem 2.1. Fix \( \tilde{b} = (1, \tilde{b}_2, \ldots, \tilde{b}_N) \in W \) and let

\[
\tilde{W} = \{ b = (1, b_2, \ldots, b_N) \in W : b_j \leq \tilde{b}_j, \; j = 2, \ldots, N \}.
\]

\( \tilde{W} \) is compact. Let \( d : \tilde{W} \to \mathbb{R} \) be given by \( d(b) = 1 + b_2 + \cdots + b_N \); \( d \) attains its minimum value in \( \tilde{W} \) at a point \( b^* = (1, b^*_2, \ldots, b^*_N) \) (notice that the minimum is strictly positive by Lemma 2.2(b)). We prove that \( \mathcal{R}(b^*) \) is the refractor that solves the problem. By conservation of energy it is enough to show that

\[
\int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_i)} f(x) \, dx = g_j \quad \text{for} \; j = 2, \ldots, N.
\]

Since \( b^* \in W \), we have \( \int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_i)} f(x) \, dx \leq g_j \) for \( j = 2, \ldots, N \). Suppose by contradiction that this inequality is strict for some \( j \), suppose for example that

\[
(2.4) \quad \int_{\mathcal{T}_{\mathcal{R}(b^*)}(m_j)} f(x) \, dx < g_2.
\]

Let \( 0 < \lambda < 1 \) and \( b^*_\lambda = (1, \lambda b^*_2, b^*_3, \ldots, b^*_N) \).

We claim that

\[
(2.5) \quad \mathcal{T}_{\mathcal{R}(b^*_\lambda)}(m_i) \setminus \text{set of measure zero} \subset \mathcal{T}_{\mathcal{R}(b^*)}(m_i)
\]

for \( i = 3, 4, \ldots, N \) and all \( 0 < \lambda < 1 \). Indeed, if \( x_0 \in \mathcal{T}_{\mathcal{R}(b^*_\lambda)}(m_i) \) is not a singular point of \( \mathcal{R}(b^*_\lambda) \), then there is a unique ellipsoid \( \frac{a}{1 - \kappa x \cdot m_i} \) that supports \( \mathcal{R}(b^*_\lambda) \) at \( x_0 \) for some \( a > 0 \). Since \( \mathcal{R}(b^*_\lambda)(x) = \min_{i \leq j \leq N} \frac{(b^*_\lambda)_j}{1 - \kappa x \cdot m_i} \), there exists \( 1 \leq j \leq N \) such that

\[
\mathcal{R}(b^*_\lambda)(x_0) = \frac{(b^*_\lambda)_j}{1 - \kappa x \cdot m_j}.
\]

That is, the ellipsoid \( \frac{(b^*_\lambda)_j}{1 - \kappa x \cdot m_j} \) supports \( \mathcal{R}(b^*_\lambda) \) at \( x_0 \).

Therefore, \( \frac{a}{1 - \kappa x \cdot m_i} = \frac{(b^*_\lambda)_j}{1 - \kappa x \cdot m_j} \) implying \( j = i \) and so \( a = (b^*_\lambda)_i = b^*_i \). Since \( \frac{b^*_i}{1 - \kappa x \cdot m_i} \geq \mathcal{R}(b^*)(x) \geq \mathcal{R}(b^*_\lambda)(x) \) for all \( x \), it follows that \( \frac{b^*_i}{1 - \kappa x \cdot m_i} \) is a supporting ellipsoid to \( \mathcal{R}(b^*) \) at \( x_0 \). This proves the claim.

This implies that

\[
\int_{\mathcal{T}_{\mathcal{R}(b^*_\lambda)(m_i)}} f(x) \, dx \leq \int_{\mathcal{T}_{\mathcal{R}(b^*)(m_i)}} f(x) \, dx \leq g_i
\]

for \( i = 3, 4, \ldots, N \) and all \( 0 < \lambda < 1 \).
Finally from Lemma 2.4, inequality (2.4) holds for all \( \lambda \) sufficiently close to one, and therefore the point \( b^*_\lambda \in \tilde{W} \) for all \( \lambda \) close to one. This is a contradiction because \( d(b^*_\lambda) < d(b^*) \). 

\[ \square \]

**Remark 2.5.** Notice that the solution in Theorem 2.1 has the form given by formula (2.3), where \( b_1 = 1 \) and \( (1, b_2, \cdots, b_N) \in W \). So from Lemma 2.2(b), we have \( b_i > 1/(1 + \kappa) \) for \( i = 2, \cdots, N \). This implies that \( \inf_{\Omega} \rho(x) = \alpha > 0 \).

### 2.2. Solution in the general case.

**Lemma 2.6.** Let \( \mathcal{R} = \{\rho(x) : x \in \overline{\Omega}\} \) be a refractor from \( \overline{\Omega} \) to \( \overline{\Omega}^\ast \) such that \( \inf_{x \in \overline{\Omega}} \rho(x) = 1 \). Then there is a constant \( C \), depending only on \( \kappa \), such that

\[ \sup_{x \in \overline{\Omega}} \rho(x) \leq C. \]

**Proof.** Suppose \( \inf_{x \in \overline{\Omega}} \rho(x) \) is attained at \( x_0 \in \overline{\Omega} \), and let \( E(m, b) \) be a supporting semi-ellipsoid to \( \mathcal{R} \) at \( \rho(x_0)x_0 \). Then

\[ 1 = \rho(x_0) = \frac{b}{1 - \kappa m \cdot x_0} \quad \text{and} \quad \rho(x) \leq \frac{b}{1 - \kappa m \cdot x} \quad \forall x \in \overline{\Omega}. \]

From the first equation we get that \( b \leq 1 + \kappa \), and using this in the inequality we obtain

\[ \rho(x) \leq \frac{1 + \kappa}{1 - \kappa^2} \quad \text{for all} \ x \in \overline{\Omega} \]

which proves the lemma. \[ \square \]

**Theorem 2.7.** Let \( f \in L^1(\overline{\Omega}) \) with \( \inf_{\overline{\Omega}} f > 0 \), and let \( \mu \) be a Radon measure on \( \overline{\Omega}^\ast \), such that

\[ \int_{\overline{\Omega}} f(x) \, dx = \mu(\overline{\Omega}^\ast) \]

Then there exists a weak solution \( \mathcal{R} \) of the refractor problem for the case \( \kappa < 1 \), with emitting illumination intensity \( f \) and prescribed refracted illumination intensity \( \mu \).
Proof. Fix $l \in \mathbb{N}, l \geq 2$. Partition $\overline{\Omega}^r$ into a finite number of disjoint Borel subsets $\omega_i^l, \ldots, \omega_k^l$ such that $\text{diam}(\omega_i^l) \leq \frac{1}{l}$. Choose points $m_i^l \in \omega_i^l$ and define a measure on $\overline{\Omega}^r$

$$\mu_l = \sum_{i=1}^{k_l} \mu(\omega_i^l) \delta_{m_i^l}.$$ 

Then

$$\mu_l(\overline{\Omega}^r) = \sum_{i=1}^{k_l} \mu(\omega_i^l) = \mu(\overline{\Omega}^r) = \int_{\overline{\Omega}^r} f(x) \, dx.$$ 

If $h \in C(\overline{\Omega}^r)$, then

$$\int_{\overline{\Omega}^r} h \, d\mu_l - \int_{\overline{\Omega}^r} h \, d\mu = \sum_{i=1}^{k_l} \left( \int_{\omega_i^l} h(x) \, d\mu_l - \int_{\omega_i^l} h(x) \, d\mu \right)$$

$$= \sum_{i=1}^{k_l} \left( \int_{\omega_i^l} h(m_i^l) \, d\mu - \int_{\omega_i^l} h(x) \, d\mu \right)$$

$$= \sum_{i=1}^{k_l} \int_{\omega_i^l} (h(m_i^l) - h(x)) \, d\mu.$$ 

Since $h \in C(\overline{\Omega}^r)$ and $\text{diam}(\omega_i^l) < \frac{1}{l}$, we obtain that

$$\int_{\overline{\Omega}^r} h \, d\mu_l \to \int_{\overline{\Omega}^r} h \, d\mu \quad \text{as } l \to \infty$$

and hence $\mu_l$ converges weakly to $\mu$.

By Theorem 2.1 let $\mathcal{R}_l = \{\rho_l(x) : x \in \overline{\Omega}\}$ be the solution corresponding to $\mu_l$, that is,

$$\mathcal{M}_{\mathcal{R}_l,f}(\omega) = \mu_l(\omega)$$

for every Borel subset $\omega$ of $\overline{\Omega}^r$. Notice that from Remark 2.5 $\inf_{\Omega} \rho_l(x) = \alpha_l > 0$. In view of Remark 1.8 the refractor defined by the function $\frac{\rho_l(x)}{\alpha_l}$ solves the same problem and $\inf_{\Omega} \frac{\rho_l(x)}{\alpha_l} = 1$. So by normalize $\rho_l$, we may assume that $\inf_{\Omega} \rho_l(x) = 1$. 

Then by Lemma (2.6) there exists a uniform bound \( C = C(\kappa) \) such that

\[
\sup_{x \in \Omega_l} \rho_l(x) \leq C \quad \text{for all } l \geq 1.
\]

Also if \( x_o, x_1 \in \overline{\Omega} \) and \( E(m_o, b_o) \) is a supporting semi ellipsoid to \( R_l \) at \( \rho_l(x_o)x_o \) then for \( x_1 \in \overline{\Omega} \) we have

\[
\rho_l(x_1) - \rho_l(x_o) \leq \frac{b_o}{1 - \kappa m_o \cdot x_1} - \frac{b_o}{1 - \kappa m_o \cdot x_o} = \frac{\kappa b_0 m_o \cdot (x_1 - x_0)}{(1 - \kappa m_o \cdot x_1)(1 - \kappa m_o \cdot x_o)} \leq \frac{\kappa b_0 |m_o| |x_1 - x_0|}{(1 - \kappa m_o \cdot x_1)(1 - \kappa m_o \cdot x_o)} \leq C \frac{\kappa}{1 - \kappa} |x_1 - x_o|.
\]

By changing the roles of \( x_o \) and \( x_1 \) we conclude that

\[
|\rho_l(x_1) - \rho_l(x_o)| \leq C \frac{\kappa}{1 - \kappa} |x_1 - x_o| \quad \text{for all } l \geq 1.
\]

Thus \( \{\rho_l : l \geq 1\} \) is an equicontinuous family which is bounded uniformly. Then by Arzelà - Ascoli Theorem, if need be by taking subsequence, we have that \( \rho_l \to \rho \) uniformly on \( \overline{\Omega} \). By Lemma 1.5(i), \( \mathcal{R} = \{\rho(x) : x \in \overline{\Omega}\} \) is a refractor.

We claim that \( \mathcal{M}_{\mathcal{R},f} \) converges weakly to \( \mathcal{M}_{\mathcal{R},f} \). Indeed, if \( F \) is any closed subset of \( \overline{\Omega^*} \) then by Lemma 1.5(ii) and reverse Fatou we have

\[
\limsup_{l \to \infty} \mathcal{M}_{\mathcal{R}_l,f}(F) \leq \int \limsup_{l \to \infty} T_{\mathcal{R}_l}(F) f(x) \, dx \leq \int T_{\mathcal{R}(F)} f(x) \, dx = \mathcal{M}_{\mathcal{R},f}(F).
\]

Moreover for any open set \( G \subset \overline{\Omega^*} \) we claim that

\[
(2.6) \quad \mathcal{M}_{\mathcal{R}_l,f}(G) = \int_{T_{\mathcal{R}}(G)} f(x) \, dx \leq \liminf_{l \to \infty} \mathcal{M}_{\mathcal{R}_l,f}(G).
\]

Indeed, from Lemma 1.5(iii) we have

\[
\mathcal{M}_{\mathcal{R}_l,f}(G) = \int_{T_{\mathcal{R}}(G)} f(x) \, dx \leq \int \liminf_{l \to \infty} T_{\mathcal{R}_l}(G) f(x) \, dx
\]

\[
= \int_{\mathcal{R}} \liminf_{l \to \infty} \chi_{T_{\mathcal{R}_l}(G)}(x) f(x) \, dx
\]

\[
\leq \liminf_{l \to \infty} \int_{\mathcal{R}} \chi_{T_{\mathcal{R}_l}(G)}(x) f(x) \, dx = \liminf_{l \to \infty} \mathcal{M}_{\mathcal{R}_l,f}(G),
\]

by Fatou’s lemma. Consequently \( \mathcal{M}_{\mathcal{R}_l,f} \to \mathcal{M}_{\mathcal{R},f} \) weakly.
Since $\mathcal{M}_{\mathcal{R},f}(\omega) = \mu_i(\omega)$ for each Borel set $\omega$, it follows by uniqueness of the weak limit that $\mathcal{M}_{\mathcal{R},f} = \mu$. □

3. Uniqueness (discrete case)

Recall that $(b_1, \cdots, b_N)$ are positive numbers, $m_1, \cdots, m_N$ are different points in the sphere $S^{n-1}$, and $\Omega \subset S^{n-1}$ with $\inf_{x \in \overline{\Omega}, 1 \leq i \leq N} x \cdot m_i \geq \kappa$. We let

$$\rho(x) = \min_{1 \leq i \leq N} \frac{b_i}{1 - \kappa x \cdot m_i},$$

and let $\mathcal{R} = \mathcal{R}(b) = \{\rho(x)x : x \in \overline{\Omega}\}$.

**Lemma 3.1.** Suppose that the set $\mathcal{T}_\mathcal{R}(m_i)$ has positive measure. If $x_0 \in \mathcal{T}_\mathcal{R}(m_i)$, then the semi-ellipsoid $E(m_i, b_j)$ supports $\mathcal{R}$ at the point $x_0$.

**Proof.** We have $m_j \in \mathcal{N}_\mathcal{R}(x_0)$, that is, there exists a supporting semi-ellipsoid $E(m_j, b)$ to $\mathcal{R}$ at the point $x_0$ for some $b > 0$. We claim that $b = b_j$, and therefore $\frac{b_j}{1 - \kappa m_j \cdot x}$ supports $\mathcal{R}$ at $x_0$. Since $E(m_j, b)$ supports $\mathcal{R}$, we have $\rho(x) \leq \frac{b}{1 - \kappa x \cdot m_j}$ for all $x \in \overline{\Omega}$ with equality at $x = x_0$. Hence $\frac{b}{1 - \kappa x_0 \cdot m_j} \leq \frac{b_j}{1 - \kappa x_0 \cdot m_j}$, and so $b \leq b_j$. If $b = b_j$ we are done. If $b < b_j$, then

$$\rho(x) \leq \frac{b}{1 - \kappa x \cdot m_j} < \frac{b_j}{1 - \kappa x \cdot m_j}, \quad \forall x \in \overline{\Omega},$$

so $\rho(x) = \min_{k \neq j} \frac{b_k}{1 - \kappa m_k \cdot x}$, and therefore $\mathcal{R}$ cannot refract in the direction $m_j$ (except on a set of directions with measure zero) and so $\mathcal{T}_\mathcal{R}(m_j)$ has measure zero, a contradiction. □

**Lemma 3.2.** Let $\mathcal{R}_b, \mathcal{R}_{b'}$ be two solutions from Theorem 2.1 with $b = (b_1, \cdots, b_N)$, and $b' = (b'_1, \cdots, b'_N)$. Assume that $f > 0$ a.e. in $\Omega$. If $b'_1 \leq b_1$, then $b'_i \leq b_i$ for all $1 \leq i \leq N$. In particular, if $b'_1 = b_1$, then $b'_i = b_i$ for all $1 \leq i \leq N$. 
Proof. Let \( J = \{ j : b_j < b^*_j \} \) and \( I = \{ i : b_i \geq b^*_i \} \). Suppose by contradiction that \( J \neq \emptyset \).

We have \( I \neq \emptyset \) since \( 1 \in I \). Fix \( j \in J \), we have \( \frac{b_j}{1 - \kappa z \cdot m_j} < \frac{b^*_j}{1 - \kappa z \cdot m_j} \) for all \( z \in \Omega \) since \( b_j < b^*_j \). And also \( \frac{b^*_i}{1 - \kappa z \cdot m_i} \leq \frac{b_i}{1 - \kappa z \cdot m_i} \) for all \( i \in I \) and all \( z \in \overline{\Omega} \). Fix \( j \in J \) and let \( x \in \mathcal{T}_{\mathcal{R}_{b^*}}(m_j) \). Since \( \mathcal{R}_{b^*} \) is a solution to the discrete problem and \( g_i > 0 \) for all \( 1 \leq i \leq N \), we have that \( \mathcal{T}_{\mathcal{R}_{b^*}}(m_j) \) has positive measure. So from Lemma 3.1, the ellipsoid \( \frac{b^*_j}{1 - \kappa m_j \cdot z} \) supports \( \mathcal{R}_{b^*} \) at the point \( x \). Since the function defining \( \mathcal{R}_{b^*} \) is given by \( \rho'(z) = \min_{1 \leq i \leq N} \frac{b_i}{1 - \kappa m_i \cdot z} \), and \( \rho'(x) = \frac{b^*_j}{1 - \kappa m_j \cdot x} \), we therefore obtain

\[
\frac{b_j}{1 - \kappa m_j \cdot x} < \frac{b^*_j}{1 - \kappa m_j \cdot x} \leq \frac{b^*_i}{1 - \kappa m_i \cdot x} \leq \frac{b_i}{1 - \kappa m_i \cdot x}, \quad \forall i \in I.
\]

Hence by continuity, there exists \( N_x \) an open neighborhood of \( x \) such that

\[
\frac{b_j}{1 - \kappa m_j \cdot y} < \frac{b^*_j}{1 - \kappa m_j \cdot y} \quad \forall i \in I, \quad \forall y \in N_x.
\]

Since the function defining \( \mathcal{R}_b \) is \( \rho(z) = \min_{1 \leq i \leq N} \frac{b_i}{1 - \kappa m_i \cdot z} \), we get for \( y \in N_x \) that \( \rho(y) = \min_{j \in J} \frac{b_j}{1 - \kappa m_j \cdot y} \), that is, \( \rho(y) = \frac{b_{j'}}{1 - \kappa m_{j'} \cdot y} \) for some \( j' \in J \) (depending also on \( y \)) which means that \( \frac{b_{j'}}{1 - \kappa m_{j'} \cdot y} \) is a supporting ellipsoid to \( \mathcal{R}_b \) at \( y \). Therefore

\[
N_x \subset \mathcal{T}_{\mathcal{R}_b}(\bigcup_{j \in I} m_j).
\]

We then have that every point \( x \in \mathcal{T}_{\mathcal{R}_{b^*}}(\bigcup_{j \in I} m_j) \) has a neighborhood contained in \( \mathcal{T}_{\mathcal{R}_b}(\bigcup_{j \in I} m_j) \), that is,

\[
\mathcal{T}_{\mathcal{R}_{b^*}}(\bigcup_{j \in I} m_j) \subset \left( \mathcal{T}_{\mathcal{R}_b}(\bigcup_{j \in I} m_j) \right)^{\circ} \neq \overline{\Omega}.
\]

Since \( \overline{\Omega} \) is connected and \( \mathcal{T}_{\mathcal{R}_{b^*}}(\bigcup_{j \in I} m_j) \) is closed, we get that \( \left( \mathcal{T}_{\mathcal{R}_b}(\bigcup_{j \in I} m_j) \right)^{\circ} \setminus \mathcal{T}_{\mathcal{R}_{b^*}}(\bigcup_{j \in I} m_j) \) is a non empty open set. This is a contradiction with the fact that

\[
\int_{\mathcal{T}_{\mathcal{R}_b}(\bigcup_{j \in I} m_j)} f(x) \, dx = \sum_{j \in I} f_j = \int_{\mathcal{T}_{\mathcal{R}_{b^*}}(\bigcup_{j \in I} m_j)} f(x) \, dx,
\]

since \( f > 0 \) a.e..
From the lemma we deduce the uniqueness up to dilations in the discrete case. Let \( \lambda > 0 \). Notice that if \( E(m, b) \) is a supporting ellipsoid to the refractor \( \mathcal{R}_\lambda \) with defining function \( \lambda \rho(x) \) at \( x_0 \) if and only if \( E(m, b/\lambda) \) is a supporting ellipsoid to the refractor \( \mathcal{R} \) with defining function \( \rho(x) \) at the point \( x_0 \). This implies that \( \mathcal{N}_{\mathcal{R}_\lambda}(x_0) = \mathcal{N}_\mathcal{R}(x_0) \), and consequently \( \mathcal{T}_{\mathcal{R}_\lambda}(m) = \mathcal{T}_\mathcal{R}(m) \). Therefore, if \( \mathcal{R} \) is a refractor solving the problem in Theorem 2.1 then \( \mathcal{R}_\lambda \) solves the same problem for any \( \lambda > 0 \). We now prove the uniqueness. Suppose \( \mathcal{R}_b \) and \( \mathcal{R}_{b^*} \) are two solutions as in Theorem 2.1. Pick \( \lambda \) such that \( \lambda b_1 = b^*_1 \). The refractor \( \mathcal{R}_{\lambda b} \) is also a solution to Theorem 2.1, and by Lemma 3.2 we obtain that \( \lambda b_i = b^*_i \) for all \( i \). This means that \( \mathcal{R}_b \) and \( \mathcal{R}_{b^*} \) are multiples one of each other and we obtain the uniqueness.

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