NOTES ON SCHAUDER ESTIMATES

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**Lemma 1.** If $\Delta u \geq -f$ in $B_r(y)$, then

$$u(x) \leq \sup_{\partial B_r(y)} u + \frac{r^2 - |x - y|^2}{2n} \sup_{B_r(y)} f, \quad x \in B_r(y).$$

**Proof.** Let $g(x) = u(x) - \sup_{\partial B_r(y)} u - \frac{r^2 - |x - y|^2}{2n} \sup_{B_r(y)} f$. We have $\Delta g = \Delta u + \sup_{B_r(y)} f \geq -f + \sup_{B_r(y)} f \geq 0$, that is, $g$ is subharmonic in $B_r(y)$. Then $\sup_{B_r(y)} g = 0$, so $g \leq 0$ in $B_r(y)$ and the lemma follows. □

**Lemma 2.** If $u$ is a solution to $\Delta u = f$ in $B_r(y)$ and $v$ solves $\Delta v = 0$ and $v = u$ on $\partial B_r(y)$, then

$$r^2 - |x - y|^2 \frac{1}{2n} \inf_{B_r(y)} f \leq v(x) - u(x) \leq r^2 - |x - y|^2 \frac{1}{2n} \sup_{B_r(y)} f,$$

in particular,

$$|u(x) - v(x)| \leq \frac{r^2 - |x - y|^2}{2n} \sup_{B_r(y)} |f|,$$

for all $x \in B_r(y)$.

**Proof.** Since $\Delta(v - u) = -f$, the inequality immediately follows from Lemma 1 □

**Lemma 3.** Let $0 < \alpha < 1$. There exist positive constants $C_0, \epsilon_0$ and $0 < \lambda < 1$ such that for any $f$ and any solution to $\Delta u = f$ in $B_1(0)$ with $\|u\|_{L^\infty(B_1(0))} \leq 1$ and $\|f\|_{L^\infty(B_1(0))} \leq \epsilon_0$ there exists a second degree harmonic polynomial $q(x) = \frac{1}{2} \langle Ax, x \rangle + B \cdot x + C$ such that

(1) $|u(x) - q(x)| \leq \lambda^{2+\alpha}$, for $|x| \leq \lambda$,

and

(2) $\|A\| + |B| + |C| \leq C_0$.

**Proof.** Let $v$ be the harmonic function in the statement of Lemma 2. By the maximum principle $\sup_{B_1(0)} |v| \leq \sup_{B_1(0)} |u| \leq 1$. Since $v$ is harmonic

(3) $|D^\beta v(x)| \leq C(n, |\beta|) \sup_{B_1(0)} |v| \leq C(n, |\beta|)$, for $|x| \leq 1/2$. 


We shall prove that the second order Taylor polynomial of \( v \) about 0, \( q(x) = \frac{1}{2}(D^2v(0)x, x) + Dv(0) \cdot x + v(0) \), satisfies (1) and (2). In fact, (2) follows from (3). Also, \( \Delta q(x) = \text{trace}D^2q(x) = \text{trace}D^2v(0) = \Delta v(0) = 0 \), so \( q \) is harmonic. We have

\[
v(x) = q(x) + \frac{1}{3!}[(x \cdot D)^3v(x)]_{x=\xi},
\]

with \( \xi \) an intermediate point between 0 and \( x \). Then from Lemma 2 we get

\[
|u(x) - q(x)| \leq |u(x) - v(x)| + |v(x) - q(x)|
\]

\[
\leq \frac{1 - |x|^2}{2n} \sup_{B_1(0)} |f| + \frac{1}{3!} \left[ (x \cdot D)^3v(x) \right]_{x=\xi}
\]

\[
\leq \frac{1 - |x|^2}{2n} \sup_{B_1(0)} |f| + C_n |x|^3 \sup_{|z| \leq 1/2, |\beta| = 3} |D^\beta v(z)|, \quad \text{for } |x| \leq 1/2
\]

\[
\leq \frac{1 - |x|^2}{2n} \sup_{B_1(0)} |f| + C_n |x|^3 = I + II, \quad \text{from (3)}.
\]

We write

\[
II = C_n |x|^3 \leq C_n \lambda^3, \quad \text{if } |x| \leq \lambda
\]

\[
\leq \frac{1}{2} \lambda^{2+\alpha}
\]

if we pick \( \lambda \leq \left( \frac{1}{2C_n} \right)^{1/(1-\alpha)} \). With this value of \( \lambda \), we next want

\[
I \leq \frac{1}{2} \lambda^{2+\alpha}.
\]

If \( \epsilon_0 \leq n \lambda^{2+\alpha} \), we then have

\[
I = \frac{1 - |x|^2}{2n} \sup_{B_1(0)} |f| \leq \frac{1}{2n} \sup_{B_1(0)} |f| \leq \frac{1}{2n} \epsilon_0 \leq \frac{1}{2} \lambda^{2+\alpha}
\]

and we are done. \( \square \)

**Theorem 4.** Suppose \( u \in C^2(B_1(0)) \cap C(\bar{B}_1(0)) \), \( \Delta u = f \), and \( f \) is Hölder continuous at 0, i.e.,

\[
[f]_{\alpha,0} = \sup_{|x| \leq 1} \frac{|f(x) - f(0)|}{|x|^\alpha} < \infty.
\]

Then there exists a second degree polynomial \( p(x, 0) = \frac{1}{2}(Ax, x) + B \cdot x + C \) such that

\[
|u(x) - p(x, 0)| \leq C_1 |x|^{2+\alpha}, \quad \text{for } |x| \leq 1/2,
\]

with

\[
C_1 \leq C_0 \left( [f]_{\alpha,0} + \|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right),
\]
and

$$||A|| + |B| + |C| \leq C_0 \left( ||f||_{L^\infty(B_1)} + ||u||_{L^\infty(B_1)} \right).$$

Proof. We may assume that

(i) $f(0) = 0$,

(ii) $[f]_{a,0} + ||f||_{L^\infty(B_1)} \leq \epsilon_0$,

(iii) $||u||_{L^\infty(B_1)} \leq 1$.

Indeed, if we let

$$v(x) = u(x) - \frac{|x|^2}{2n} f(0), \quad h(x) = \epsilon_0 \frac{f(x) - f(0)}{[f]_{a,0} + 2||f||_{L^\infty(B_1)} + ||v||_{L^\infty(B_1)}}$$

and

$$\bar{u}(x) = \epsilon_0 \frac{v(x)}{[f]_{a,0} + 2||f||_{L^\infty(B_1)} + ||v||_{L^\infty(B_1)}},$$

then $\Delta \bar{u} = h$ in $B_1$, $h$ satisfies (i) and (ii), and $\bar{u}$ satisfies (iii).

Claim: there exists a sequence of harmonic polynomials $p_k(x) = \frac{1}{2} \langle A_k x, x \rangle + B_k \cdot x + C_k$ such that

$$|u(x) - p_k(x)| \leq \lambda^{(2+a)k}, \quad \text{for } |x| \leq \lambda^k$$

and

$$||A_k - A_{k+1}|| \leq C \lambda^{ak}, \quad |B_k - B_{k+1}| \leq C \lambda^{(a+1)k}, \quad |C_k - C_{k+1}| \leq C \lambda^{(a+2)k},$$

for $k = 1, \cdots$ where $C$ is a universal constant. In view of (ii) and (iii) above we can apply Lemma 3, and we let $p_1(x)$ be the polynomial in that lemma. Suppose $p_k(x)$ is constructed. We will construct $p_{k+1}$. Let

$$w(x) = \frac{(u - p_k)(\lambda^k x)}{\lambda^{(a+2)k}}.$$ 

We have

$$\Delta w(x) = \frac{1}{\lambda^{(a+2)k}} \left[ \lambda^{2k} (\Delta u)(\lambda^k x) - \lambda^{2k} (\Delta p_k)(\lambda^k x) \right] = \frac{1}{\lambda^{ak}} f(\lambda^k x) = g_k(x).$$

From (5), $||w||_{L^\infty(B_1)} \leq 1$ and from (i) and (ii) above, $||g_k||_{L^\infty(B_1)} \leq \epsilon_0$. Hence by application of Lemma 3 to $w$, we get a harmonic polynomial $q_k(x)$ -depending on $g_k$- such that

$$|w(x) - q_k(x)| \leq \lambda^{2+\alpha}, \quad \text{for } |x| \leq \lambda.$$

From the definition of $w$ and (7) we then get

$$|u(\lambda^k x) - p_k(\lambda^k x) - \lambda^{(2+\alpha)k} q_k(x)| \leq \lambda^{(2+\alpha)(k+1)}, \quad \text{for } |x| \leq \lambda.$$
Therefore, if we take
\begin{equation}
(8) \quad p_{k+1}(x) = p_k(x) + \lambda^{(2 + \alpha)k} q_k(x/\lambda^k),
\end{equation}
then \(p_{k+1}\) satisfies (5) with \(k\) replaced by \(k + 1\). Writing \(q_k(x) = \frac{1}{2} \langle A_k x, x \rangle + B_k^* x + C_k\), from (8) we get
\begin{align*}
A_{k+1} &= A_k + \lambda^{(2 + \alpha)k} A_k^* B_k = B_k + \lambda^{(2 + \alpha)k} B_k^*, \\
C_{k+1} &= C_k + \lambda^{(2 + \alpha)k} C_k^*
\end{align*}
for \(k = 1, 2, \cdots\) and then (6) follows from (2). This completes the proof of the claim.

Next, we notice that from (6), (2), and since \(0 < \lambda < 1\), it follows that \(A_k, B_k, C_k\) are Cauchy sequences and therefore we let
\[ p(x, 0) = \frac{1}{2} \langle A_{\infty} x, x \rangle + B_{\infty} \cdot x + C_{\infty}, \]
where \(A_{\infty}, B_{\infty}, C_{\infty}\) are the corresponding limits. We show that \(p(x, 0)\) satisfies (4). Given \(|x| \leq 1/2\), let \(k\) be a positive integer such that \(\lambda^{k+1} < |x| \leq \lambda^k\). Hence
\[ |p_k(x) - p(x, 0)| \leq C (\lambda^{ak} |x|^2 + \lambda^{(\alpha+1)k} |x| + \lambda^{(\alpha+2)k} |x|^{2+\alpha}). \]
and from (5) we obtain
\[ |u(x) - p(x, 0)| \leq |u(x) - p_k(x)| + |p_k(x) - p(x, 0)| \leq C |x|^{2+\alpha} + C |x|^{2+\alpha} \]
and we are done. \(\square\)

Suppose \(f \in C^\alpha(B_1(0))\) and \(u\) is a solution to \(\Delta u = f\) in \(B_1(0)\). Let \(y \in B_1(0)\) and \(r < \text{dist}(y, \partial B_1(0))\). Define \(g(x) = r^2 f(y + rx)\) and \(v(x) = u(y + rx)\) for \(x \in B_1(0)\). We have that \(v\) is a solution to \(\Delta v = g\) in \(B_1(0)\) and
\[ [g]_{a,0} = \sup_{|x| \leq 1} \frac{|g(x) - g(0)|}{|x|^a} = r^2 \sup_{|x| \leq 1} \frac{|f(y + rx) - f(y)|}{|x|^a} = r^{2+\alpha} \sup_{|z| \leq r} \frac{|f(y + z) - f(y)|}{|z|^a}. \]
From Theorem 4 applied to \(v\), there exists a quadratic polynomial \(p(x, 0)\) such that
\begin{equation}
(9) \quad |v(x) - p(x, 0)| \leq C_1 |x|^{2+\alpha}, \quad \text{for } |x| \leq 1/2,
\end{equation}
with
\begin{equation}
(10) \quad C_1 \leq C_0 \left( [g]_{a,0} + \|g\|_{L^\infty(B_1(0))} + \|v\|_{L^\infty(B_1(0))} \right).
\end{equation}
From (9) and the definition of \(v\) we get that
\[ |u(z) - p((z - y)/r, 0)| \leq \frac{C_1}{r^{2+\alpha}} |z - y|^{2+\alpha}, \quad \text{for } |z - y| \leq r/2. \]
We have $\|g\|_{L^\infty(B_1(0))} = r^2 \|f\|_{L^\infty(B_1(0))}$ and $\|v\|_{L^\infty(B_1(0))} = \|u\|_{L^\infty(B_1(0))}$. If we let $q(x,y) = p((x-y)/r,0)$, and $r = \text{dist}(y,\partial B_1(0))$, then

$$|u(x) - q(x,y)| \leq C_1 |x-y|^{2+a}, \quad \text{for } |x-y| \leq \text{dist}(y,\partial B_1(0))/2,$$

with

$$C_1 = \frac{C_1}{\text{dist}(y,\partial B_1(0))^{2+a}} \leq C_0 \left( \sup_{\|z\| \leq \text{dist}(y,\partial B_1(0))} \frac{|f(y+z) - f(y)|}{|z|^a} + \text{dist}(y,\partial B_1(0))^{-a} \|f\|_{L^\infty(B_1(0))} + \text{dist}(y,\partial B_1(0))^{-2-a} \|u\|_{L^\infty(B_1(0))} \right).$$

In particular, we obtain that if $\Delta u = f$ in $B_1(0)$, then for each $y \in B_{1/2}(0)$ there exists a quadratic polynomial $p(x,y)$ such that

$$|u(x) - p(x,y)| \leq C^* |x-y|^{2+a}, \quad \text{for } |x-y| \leq 1/4,$$

with

$$C^* \leq C_0 \left( \|f\|_{L^\infty(B_1(0))} + \|f\|_{L^\infty(B_1(0))} + \|u\|_{L^\infty(B_1(0))} \right).$$

**Lemma 5.** Suppose $u \in C^2(\Omega)$ is such that there exist constants $C_1 > 0$ and $0 < \alpha < 1$ so that for each $y \in \Omega$ there exists a quadratic polynomial $p(x,y)$ such that

(11) $|u(x) - p(x,y)| \leq C_1 |x-y|^{2+a}, \quad \text{for all } x \in B(y,\text{dist}(y,\partial \Omega)/2).$

Then

(12) $p(x,y) = \frac{1}{2} (x-y)^t D^2 u(y)(x-y) + Du(y) \cdot (x-y) + u(y),$

and

(13) $|D_{ij} u(x_1) - D_{ij} u(x_2)| \leq C_1 |x_1 - x_2|^\alpha$

for all $x_1, x_2 \in \Omega$ with $\text{dist}(x_i, \partial \Omega) > \text{diam}(\Omega)/2$.

**Proof.** We use the following result: if $f \in C^2(I)$ where $I$ is an open interval, then

(14) $\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a),$

for $a \in I$. (Notice that the converse to this result is not true, take $f(x) = x|x|$ at $a = 0$).

From (11) it follows immediately that $p(y,y) = u(y)$. On the other hand,

$$D_j u(y) = \lim_{h \to 0} \frac{u(y+he_j) - u(y)}{h},$$
and
\[ \left| \frac{u(he_i + y) - u(y)}{h} - \frac{p(he_i + y, y) - p(y, y)}{h} \right| \leq \frac{C_1 |he|^{2+\alpha}}{|h|} \to 0, \]
as \( h \to 0 \). So \( D_s p(y, y) = Du(y) \). If \( \eta \) is a nonzero vector in \( \mathbb{R}^n \) and \( g_\eta(t) = u(t \eta + y) \), then \( g_\eta''(t) = \sum_{i,j=1}^n \eta_i \eta_j u_{ij}(t \eta + y) \). In particular, \( g_\alpha''(0) = u_{kk}(y) \), and \( g_{\epsilon \ell}''(0) = u_{kk}(y) + 2u_{k\ell}(y) + u_{\ell\ell}(y) \). Hence
\[ u_{k\ell}(y) = \frac{1}{2} \left( g_{\epsilon \ell}''(0) - g_\alpha''(0) - g_\alpha''(0) \right), \]
and from \[ \text{(14)} \] we get
\[ u_{k\ell}(y) = \frac{1}{2} \lim_{h \to 0} \frac{\Delta_{k\ell} u(h, y) + \Delta_{k\ell} u(-h, y) - 2\Delta_{k\ell} u(0, y)}{h^2} \]
where
\[ \Delta_{k\ell} u(h, y) = u(h(e_k + e_\ell) + y) - u(h e_k + y) - u(h e_\ell + y). \]
Also
\[ p_{k\ell}(y) = \frac{1}{2} \lim_{h \to 0} \frac{\Delta_{k\ell} p(h, y) + \Delta_{k\ell} p(-h, y) - 2\Delta_{k\ell} p(0, y)}{h^2} \]
Since
\[ |\Delta_{k\ell} u(h, y) - (p(h(e_k + e_\ell) + y, y) - p(h e_k + y, y) - p(h e_\ell + y, y))| \leq 4 C_1 |h|^{2+\alpha}, \]
\[ |\Delta_{k\ell} u(-h, y) - (p(-h(e_k + e_\ell) + y, y) - p(-h e_k + y, y) - p(-h e_\ell + y, y))| \leq 4 C_1 |h|^{2+\alpha}, \]
and \( \Delta_{k\ell} u(0, y) = -u(y) = -p(y, y) \) we obtain that \( D^2 u(y) = D^2_p p(y, y) \) and then \[ \text{(12)} \] is proved.

To prove \[ \text{(13)} \] we use the following lemma of Calderón-Zygmund, \[ \text{[CZ61, Lemma 2.6]} \].

**Lemma 6.** Given an integer \( m \geq 0 \), there exists a function \( \varphi \in C^\infty_0(\mathbb{R}^n) \) with support in the unit ball such that \( \varphi_\epsilon \ast P = P \) for each \( \epsilon > 0 \) and every polynomial \( P \) of degree \( \leq m \). As usual, \( \varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon) \).

**Proof.** Let \( C \) be the class of \( C^\infty \) functions in \( \mathbb{R}^n \) with support in the unit ball, and define
\[ T(\phi) = \int_{\mathbb{R}^n} \phi(x) x^\alpha \, dx. \]
The linear transformation \( T \) maps \( C \) into the vector space \( V \) of all points \( \{\xi_\alpha\} \) with \( 0 \leq |\alpha| \leq m \). We claim that this map is onto. Otherwise, \( T(C) \) is a subspace
strictly contained in $V$, and so $T(C)$ has an orthogonal complement $V' \neq \{0\}$ in $V$. Therefore there exists $\{\eta_a\} \in V'$ not all zero with

$$\sum \eta_a \xi_a = \int_{R^n} \phi(x) \sum \eta_a x^a \, dx = 0$$

for all $\phi \in C$. In particular, if $\psi \in C$ is any function with $\psi > 0$ for $|x| > 1$, then taking $\phi(x) = \psi(x) \sum \eta_a x^a$, we obtain

$$\int_{R^n} \psi(x) \left| \sum \eta_a x^a \right|^2 \, dx = 0$$

and consequently $|\sum \eta_a x^a| = 0$ for $|x| < 1$ and therefore $\eta_a$ are all zero, a contradiction. Therefore $T(C) = V$ and so there exists $\phi \in C$ such that $\int_{R^n} \phi(x) \, dx = 1$ and $\int_{R^n} \phi(x) x^a \, dx = 0$ for $0 < |x| \leq m$. If $Q(x)$ is a polynomial of degree $\leq m$, then $\int_{R^n} \phi(x) Q(x) \, dx = Q(0)$. Therefore, $e^{-n} \int_{R^n} \phi((x - y)/e) Q(y) \, dy = \int_{R^n} \phi(z) Q(x + e\tilde{z}) \, dz = Q(x)$. \hfill \Box

Let $x_1, x_2 \in \Omega$ be such that $\text{dist}(x_i, \partial \Omega) > \text{diam}(\Omega)/2$ and write

$$u(x) = u(x) - p(x, x_1) + p(x, x_1)$$
$$u(x) = u(x) - p(x, x_2) + p(x, x_2),$$

and convolving these expressions with $\varphi_e$ and using Lemma 6 with $m = 2$ we get

$$u_e(x) = [u - p(\cdot, x_1)] \ast \varphi_e(x) + p(x, x_1)$$
$$u_e(x) = [u - p(\cdot, x_2)] \ast \varphi_e(x) + p(x, x_2),$$

for $\text{dist}(x, \partial \Omega) > \varepsilon$, and taking derivatives

$$D_{ij} u_e(x) = [u - p(\cdot, x_1)] \ast D_{ij} \varphi_e(x) + D_{ij} u(x_1)$$
$$D_{ij} u_e(x) = [u - p(\cdot, x_2)] \ast D_{ij} \varphi_e(x) + D_{ij} u(x_2).$$

Hence

$$D_{ij} u(x_1) - D_{ij} u(x_2) = [u - p(\cdot, x_1)] \ast D_{ij} \varphi_e(x) - [u - p(\cdot, x_1)] \ast D_{ij} \varphi_e(x)$$
$$= e^{-n-2} \int_{|y-x|<\varepsilon} [u(y) - p(y, x_2)] D_{ij} \varphi((x - y)/e) \, dy$$
$$- e^{-n-2} \int_{|y-x|<\varepsilon} [u(y) - p(y, x_1)] D_{ij} \varphi((x - y)/e) \, dy$$
$$= I - II.$$
If we let \( x = (x_1 + x_2)/2 \), and \( \epsilon = |x_1 - x_2|/2 \), we get that \( B_\epsilon(x) \subset B_{2\epsilon}(x_i) \) for \( i = 1, 2 \), and so from (11) we get

\[
|I| \leq e^{-n-2} C_1 \int_{|y - x_2| < 2\epsilon} |y - x_2|^{2+\alpha} |D_{ij}\varphi((x - y)/\epsilon)| \, dy
\]

\[
\leq e^{-n-2} C_1 \|D_{ij}\varphi\|_\infty \int_{|y - x_2| < 2\epsilon} |y - x_2|^{2+\alpha} \, dy = C_n C_1 \epsilon^\alpha = C |x_1 - x_2|^\alpha
\]

and

\[
|II| \leq e^{-n-2} C_1 \int_{|y - x_1| < 2\epsilon} |y - x_1|^{2+\alpha} |D_{ij}\varphi((x - y)/\epsilon)| \, dy
\]

\[
\leq e^{-n-2} C_1 \|D_{ij}\varphi\|_\infty \int_{|y - x_1| < 2\epsilon} |y - x_1|^{2+\alpha} \, dy = C_n C_1 \epsilon^\alpha = C |x_1 - x_2|^\alpha
\]

and (13) follows. \( \square \)

**References**


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