What is microlocal analysis
and what is it good for.

Gerardo Mendoza
Temple University

Big ideas seminar
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Solvability already illustrates the role of the symplectic structure of the cotangent bundle.

This is the propaganda page

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(There is also the total symbol)

Why $\mathcal{C} = \{(x,\xi) : \xi \neq 0,\ p(x,\xi) = 0\}$ matters:

$$P(x,D_x)u = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} \Big[\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \Big] u(y)\,dy\,d\xi.$$

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But not always: not for $L = D_{x_1} + iD_{x_2} + i(x_1 + x_2)D_{x_3}$ (Hans Lewy) or for $M = D_{x_1} + ix_1 D_{x_2}$ (Mizohata operator).

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In particular if $p(x_0, \xi_0) = 0$ then $p(\gamma(t)) = 0$ on the integral curve γ through (x_0, ξ_0).

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Then P is solvable at x_0 iff $(\mathcal{P})$ holds in a neighborhood U of x_0.

(Nirenberg & Treves, Treves, Beals & Fefferman, Hörmander (Moyer), Lerner, Dencker)

Example: $L = D_{x_1} + iD_{x_2} + i(x_1 + x_2)D_{x_3}$: $\sigma(L) = \xi_1 + i\xi_2 + i(x_1 + ix_2)\xi_3$

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$\gamma(t) = (x_1 + t,\ x_2,\ x_3 - x_2 t;\ \xi_1 + \xi_3 t,\ \xi_2,\ \xi_3)$ ← general integral curve

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$\operatorname{Im}\sigma(L)(\gamma(t)) = -x_1\xi_3 + (x_1 + t)\xi_3 = \xi_3 t$ changes sign at $t = 0$

Solvability.

Definition: $P(x,D)$ solvable at $x_0 \in \Omega$: for every $f \in C^\infty(\Omega)$ (or in L^2, or...) there is u defined in a neighborhood of x_0 such that $Pu = f$. (modulo a finite dimensional space $\subset C^\infty$)

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$M = D_{x_1} + ix_1 D_{x_2}$ is not solvable at $K \Subset \mathbb{R}^2$ if $K \cap \{x_1 = 0\} \neq \emptyset$

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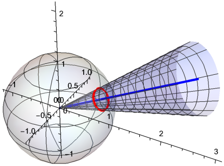

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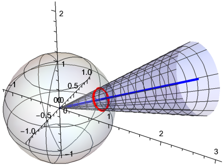

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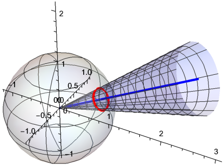

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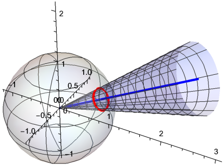

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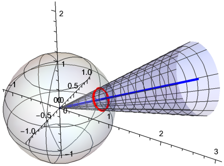

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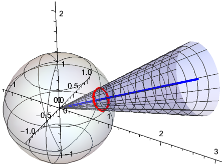

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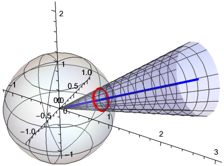

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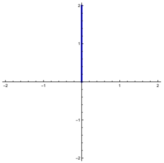

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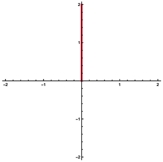

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This (and more work) implies: If M is compact and $\lambda_1, \lambda_2 \dots$ are the eigenvalues of Δ including multiplicity, then $k(t) = \sum_j e^{\imath t\sqrt{\lambda_j}}$ is defined and if $t \neq 0$ is a singularity of k then there is a closed geodesic of period t. Duistermaat-Guillemin

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$$Pu \in C^\infty \text{ implies } \mathrm{WF}(u) \subset \mathcal{C} = \{(t, x, \tau, \xi) : \tau^2 = \sum_{i,j} g^{ij}(x)\xi_i\xi_j\}$$

Propagation of singularities: If $Pu \in C^\infty$ and $\nu_0 \in WF(u)$, then the entire integral curve of $H_{\boldsymbol{\sigma}(P)}$ is contained in $\mathrm{WF}(u)$.

Special case of a theorem of Duistermaat-Hörmander

This (and more work) implies: If M is compact and $\lambda_1, \lambda_2 \ldots$ are the eigenvalues of Δ including multiplicity, then $k(t) = \sum_j e^{it\sqrt{\lambda_j}}$ is defined and if $t \neq 0$ is a singularity of k then there is a closed geodesic of period t.

Duistermaat-Guillemin

Like analysts the world over, we were amazed at how simple this subject becomes when viewed from the perspective of the cotangent bundle.

V. Guillemin, *Working single-mindedly on a project. In memoriam Hans Duistermaat*.

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$$\textstyle\sum_\alpha P_\alpha = I$$

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