On global hypoellipticity

Gerardo A. Mendoza

Temple University

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> *Functional-analytic properties of the partial differential operator have implications on the nature of the topological objects involved (the manifold or the vector bundles).*

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Euler characteristic of $\mathcal{M}$

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Principal type is meant in the more restricted sense that $d_\xi \boldsymbol{\sigma}(P)$ vanishes nowhere on $T^*\mathcal{M}\backslash 0$.

$\boldsymbol{\sigma}(P)$ is linear in the fiber variable of $T^*\mathcal{M}$;
the condition is $\boldsymbol{\sigma}(P)|_{\mathrm{Fiber}} \neq 0$

The existence of a first order differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type implies the existence of a line bundle $\mathcal{V}$ such that $F \otimes \mathcal{V} \otimes E^*$ is trivial.

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Actually, hypoellipticity + principal type + $E = F$ already implies $\mathcal{M} =$ torus.
(Hypoellipticity is needed to get $\deg(\mathcal{V}) = \pm\chi(\mathcal{M})$)

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The relevant condition is hypoellipticity, not the existence of a differential operator of a certain kind

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In the context of ellipticity...

In joint work with H. Jacobowitz[4] we showed among other things:

- If $\mathcal{M}$ is a smooth orientable manifold and $E, F \to \mathcal{M}$ are smooth complex vector bundles of the same rank, then the existence of an elliptic pseudo-differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F) \qquad (\dagger)$$

implies that

$$\mathrm{c}(F) - \mathrm{c}(E) = k\mathbf{e}_{\mathcal{M}} \qquad (\ddagger)$$

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Also

Why $\neq 0$?

- If $\mathcal{M}$ is compact without boundary, $\dim \mathcal{M} = 2$ *and* $\chi(\mathcal{M}) \neq 0$, *and if $(\dagger)$ is an elliptic pseudo-differential operator and $(\ddagger)$ holds, then $\boldsymbol{\sigma}(P)$ is homotopic through elliptic symbols, to that of an elliptic* differential *operator of order $|k|$.*

[4] H. Jacobowitz and —, *Elliptic equivalence of vector bundles*. Indiana Univ. Math. J. **51** (2002), 705–725.

I'll show:

- If $\mathcal{M}$ is a smooth orientable manifold and $E, F \to \mathcal{M}$ are smooth complex vector bundles of the same rank, then the existence of an elliptic pseudo-differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F) \qquad (\dagger)$$

implies that

$$\mathrm{c}(F) - \mathrm{c}(E) = k\mathbf{e}_{\mathcal{M}} \qquad (\ddagger)$$

for some $k \in \mathbb{Z}$.

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for some $k \in \mathbb{Z}$.

$\mathbf{e}_{\mathcal{M}}$ *is the Euler class of* $\mathcal{M}$: $\int_{\mathcal{M}} \mathbf{e}_{\mathcal{M}} = \chi(\mathcal{M})$.

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Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

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$$\begin{tikzcd} E \arrow[dr] & & F \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

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$$\begin{tikzcd} & E \arrow[d] & F \arrow[dl] \\ \dot{T}^*\mathcal{M}\backslash 0 \arrow[r, "\pi"] & \mathcal{M} & \end{tikzcd}$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$\begin{tikzcd} \pi^* E \arrow[dr] & & \pi^* F \arrow[dl] & E \arrow[dr] & & F \arrow[dl] \\ & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rrr, "\pi"] & & & \mathcal{M} & \end{tikzcd}$$

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$$\begin{tikzcd} \pi^* E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[dr] & \pi^* F \arrow[d] & E \arrow[dr] & & F \arrow[dl] \\ & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rr, "\pi"] & & \mathcal{M} & \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=small]
\mathrm{c}(\pi^* E) & & \mathrm{c}(\pi^* F) & \mathrm{c}(E) & & \mathrm{c}(F) \\
\pi^* E \arrow[rr, "\boldsymbol{\sigma}(P)"] \arrow[dr] & & \pi^* F \arrow[dl] & E \arrow[dr] & & F \arrow[dl] \\
& \dot{T}^*\mathcal{M}\backslash 0 \arrow[rrr, "\pi"] & & & \mathcal{M} &
\end{tikzcd}$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank}E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[d, equal] & \pi^*\mathrm{c}(F) \arrow[d, equal] & & \\
 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[uur, bend left] & \pi^*E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[dr] & \pi^*F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
 & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rr, "\pi"] & & \mathcal{M}
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\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[rr, "\boldsymbol{\sigma}(P)", "\approx"'] \arrow[dr] & & \pi^*F \arrow[dl] & E \arrow[dr] & & F \arrow[dl] \\
\text{ellipticity} & & \dot{T}^*\mathcal{M}\setminus 0 \arrow[rrr, "\pi"] & & & \mathcal{M} &
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$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \le n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[r, phantom, "="] \arrow[d, phantom, "\shortparallel"] & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\shortparallel"] & & \\
 & \mathrm{c}(\pi^* E) & \mathrm{c}(\pi^* F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^* E \arrow[r, "\boldsymbol{\sigma}(P)", "\approx"'] \arrow[dr] & \pi^* F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
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\end{tikzcd}
$$

Gysin sequence:

$$\cdots \to H^{2q-n}(\mathcal{M}) \xrightarrow{\mathbf{e}_{\mathcal{M}}} H^{2q}(\mathcal{M}) \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \le n$.

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\end{tikzcd}
$$

Gysin sequence:

$$
\cdots \to H^{2q-n}(\mathcal{M}) \xrightarrow{\mathbf{e}_{\mathcal{M}}} \underset{\mathrm{c}_q(F) - \mathrm{c}_q(E)}{H^{2q}(\mathcal{M})} \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots
$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

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 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[uur, bend left] & \pi^*E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[dr] \arrow[r, phantom, shift right=3, "\approx"] & \pi^*F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
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$$\mathrm{c}_q(F) - \mathrm{c}_q(E) \qquad \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big)$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

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 & \mathrm{c}(\pi^*E) & & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & & \mathrm{c}(F) \\
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 & \mathrm{c}(\pi^*E) & & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[rr, "\boldsymbol{\sigma}(P)"] \arrow[ddr] & & \pi^*F \arrow[ddl] & E \arrow[ddr] & & F \arrow[ddl] \\
\text{ellipticity} \arrow[rr, bend right] & & \approx & & & & \\
 & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rrr, "\pi"] & & & \mathcal{M} &
\end{tikzcd}$$

Gysin sequence:

$$\cdots \to H^{2q-n}(\mathcal{M}) \xrightarrow{\mathbf{e}_{\mathcal{M}}} H^{2q}(\mathcal{M}) \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots$$

$$\begin{array}{ccc}
\eta & \mathrm{c}_q(F) - \mathrm{c}_q(E) & \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) \\
 & \shortparallel & \shortparallel \\
 & \mathbf{e}_{\mathcal{M}} \cup \eta & 0
\end{array}$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[rr, phantom, "="] \arrow[d, phantom, "\parallel"] & & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\parallel"] & & & \\
 & \mathrm{c}(\pi^*E) & & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[rr, "\boldsymbol{\sigma}(P)", "\approx"'] \arrow[dr] & & \pi^*F \arrow[dl] & E \arrow[dr] & & F \arrow[dl] \\
\text{ellipticity} & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rrr, "\pi"] & & & \mathcal{M} &
\end{tikzcd}
$$

Gysin sequence:

$$
\begin{array}{ccccccccc}
\cdots \to & H^{2q-n}(\mathcal{M}) & \xrightarrow{\mathbf{e}_{\mathcal{M}}} & H^{2q}(\mathcal{M}) & \xrightarrow{\pi^*} & H^{2q}(S^*\mathcal{M}) & \xrightarrow{\pi_*} & H^{2q-n+1}(\mathcal{M}) & \to \cdots \\
 & \parallel & & & & & & & \\
 & 0 \text{ if } 2q < n & & \mathrm{c}_q(F) - \mathrm{c}_q(E) & & \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) & & & \\
 & & & \parallel & & \parallel & & & \\
 & & & \mathbf{e}_{\mathcal{M}} \cup \eta & & 0 & & &
\end{array}
$$

Proof: $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$, $c_q(E) \in H^{2q}(\mathcal{M})$, $r = \operatorname{rank} E$, $2r \leq n$.

For E of rank 1: $c_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{array}{ccccc}
\pi^* c(E) & = & \pi^* c(F) & & \\
\shortparallel & & \shortparallel & & \\
c(\pi^* E) & & c(\pi^* F) & c(E) & c(F)
\end{array}
$$

Axiom

$$
\begin{tikzcd}
\pi^* E \arrow[rr, "\boldsymbol{\sigma}(P)"] \arrow[dr] \arrow[rr, phantom, shift right=3, "\approx"] & & \pi^* F \arrow[dl] & E \arrow[dr] & & F \arrow[dl] \\
& \dot{T}^*\mathcal{M}\backslash 0 \arrow[rrr, "\pi"] & & & \mathcal{M} &
\end{tikzcd}
$$

ellipticity

Gysin sequence:

$$
\cdots \to \underset{\substack{\shortparallel \\ 0 \text{ if } 2q < n}}{H^{2q-n}(\mathcal{M})} \xrightarrow{\mathbf{e}_{\mathcal{M}}} H^{2q}(\mathcal{M}) \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots
$$

$$
\begin{array}{cc}
c_q(F) - c_q(E) & \pi^*\big(c_q(F) - c_q(E)\big) \\
\shortparallel & \shortparallel \\
\mathbf{e}_{\mathcal{M}} \cup \eta & 0 \\
\shortparallel & \\
0 \text{ if } 2q < n &
\end{array}
$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

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\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\parallel"] & & \\
 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[r, phantom, "\approx"', shift right=2] \arrow[dr] & \pi^*F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
\text{ellipticity} & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rr, "\pi"] & & \mathcal{M}
\end{tikzcd}
$$

Gysin sequence:

$$
\begin{tikzcd}
\cdots \arrow[r] & H^{2q-n}(\mathcal{M}) \arrow[r, "\mathbf{e}_{\mathcal{M}}"] \arrow[d, phantom, "\parallel"] & H^{2q}(\mathcal{M}) \arrow[r, "\pi^*"] & H^{2q}(S^*\mathcal{M}) \arrow[r, "\pi_*"] & H^{2q-n+1}(\mathcal{M}) \arrow[r] & \cdots \\
 & 0 \text{ if } 2q < n & \mathrm{c}_q(F) - \mathrm{c}_q(E) \arrow[d, phantom, "\parallel"] & \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) \arrow[d, phantom, "\parallel"] & & \\
 & & \mathbf{e}_{\mathcal{M}} \cup \eta \arrow[d, phantom, "\parallel"] & 0 & & \\
 & & 0 \text{ if } 2q < n & & &
\end{tikzcd}
$$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = 0 \text{ if } 2q < n$$

Proof: $\quad \mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

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 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
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\end{tikzcd}$$

Gysin sequence:

$$\cdots \to \underset{\substack{\shortparallel \\ \mathbb{Z} \text{ if } 2q = n}}{H^{2q-n}(\mathcal{M})} \xrightarrow{\mathbf{e}_{\mathcal{M}}} H^{2q}(\mathcal{M}) \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots$$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = \mathbf{e}_{\mathcal{M}} \cup \eta \qquad \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) = 0$$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = 0 \text{ if } 2q < n$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \mathrm{rank} E$, $2r \leq n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[r, phantom, "="] \arrow[d, phantom, "\shortparallel"] & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\shortparallel"] & & \\
 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[dr] \arrow[r, phantom, shift right=3ex, "\approx"] & \pi^*F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
\text{ellipticity} & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rr, "\pi"] & & \mathcal{M}
\end{tikzcd}
$$

Gysin sequence:

$$\cdots \to H^{2q-n}(\mathcal{M}) \xrightarrow{\mathbf{e}_{\mathcal{M}}} H^{2q}(\mathcal{M}) \xrightarrow{\pi^*} H^{2q}(S^*\mathcal{M}) \xrightarrow{\pi_*} H^{2q-n+1}(\mathcal{M}) \to \cdots$$

$H^{2q-n}(\mathcal{M}) = \mathbb{Z}$ if $2q = n$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = \mathbf{e}_{\mathcal{M}} \cup \eta = k\mathbf{e}_{\mathcal{M}} \text{ if } 2q = n$$

$$\pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) = 0$$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = 0 \text{ if } 2q < n$$

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \operatorname{rank} E$, $2r \le n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

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 & \pi^*\mathrm{c}(E) \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\parallel"] & & \\
 & \mathrm{c}(\pi^*E) & \mathrm{c}(\pi^*F) & \mathrm{c}(E) & \mathrm{c}(F) \\
\text{Axiom} \arrow[ur, bend left] & \pi^*E \arrow[r, "\boldsymbol{\sigma}(P)"] \arrow[r, phantom, shift right=3, "\approx"'] \arrow[dr] & \pi^*F \arrow[d] & E \arrow[dr] & F \arrow[d] \\
\text{ellipticity} & & \dot{T}^*\mathcal{M}\backslash 0 \arrow[rr, "\pi"] & & \mathcal{M}
\end{tikzcd}$$

Gysin sequence:

$$\begin{array}{ccccccccc}
\cdots \to & H^{2q-n}(\mathcal{M}) & \xrightarrow{\mathbf{e}_{\mathcal{M}}} & H^{2q}(\mathcal{M}) & \xrightarrow{\pi^*} & H^{2q}(S^*\mathcal{M}) & \xrightarrow{\pi_*} & H^{2q-n+1}(\mathcal{M}) & \to \cdots \\
 & \parallel & & & & & & & \\
 & \mathbb{Z} \text{ if } 2q = n & & \mathrm{c}_q(F) - \mathrm{c}_q(E) & & \pi^*\big(\mathrm{c}_q(F) - \mathrm{c}_q(E)\big) & & & \\
 & & & \parallel & & \parallel & & & \\
 & & & \mathbf{e}_{\mathcal{M}} \cup \eta & & 0 & & & \\
 & & & \parallel & & & & & \\
 & & & k\mathbf{e}_{\mathcal{M}} \text{ if } 2q = n & & & & &
\end{array}$$

$\mathrm{c}_q(F) - \mathrm{c}_q(E) = 0$ if $2q < n$ and $\mathrm{c}_q(F) - \mathrm{c}_q(E) = k\mathbf{e}_{\mathcal{M}}$ if $2q = n$.

Proof: $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots + \mathrm{c}_r(E)$, $\mathrm{c}_q(E) \in H^{2q}(\mathcal{M})$, $r = \operatorname{rank} E$, $2r \le n$.

For E of rank 1: $\mathrm{c}_1(E) = \frac{i}{2\pi}$curvature of E

$$
\begin{tikzcd}
 & \pi^*\mathrm{c}(E) \arrow[r, phantom, "="] \arrow[d, phantom, "\shortparallel"] & \pi^*\mathrm{c}(F) \arrow[d, phantom, "\shortparallel"] & & \\
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$H^{2q-n}(\mathcal{M}) = \mathbb{Z}$ if $2q = n$

$$\mathrm{c}_q(F) - \mathrm{c}_q(E) = \mathbf{e}_{\mathcal{M}} \cup \eta = k\mathbf{e}_{\mathcal{M}} \text{ if } 2q = n$$

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First order differential operators of principal type

There is $P \in \mathrm{Diff}^1(\mathcal{M}; E, F)$ *of principal type iff there is a line subbundle* $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ *such that* $F \otimes \mathcal{V} \otimes E^*$ *is trivial.*

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$$P(h\phi) = (Lh + ch)\psi$$

L is a vector field on U and c a function

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L is a local frame of $\mathcal{V}$.

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Since the section vanishes nowhere, the line bundle $F \otimes \mathcal{V} \otimes E^*$ is trivial.

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$$\deg E - \deg F = \deg \mathcal{V}.$$

$0 = \mathrm{c}_1(F \otimes \mathcal{V} \otimes E^*) = \mathrm{c}_1(F) + \mathrm{c}_1(\mathcal{V}) - \mathrm{c}_1(E)$

$\deg E = \int_M \mathrm{c}_1(E)$

Since $F \otimes \mathcal{V} \otimes E^*$ is trivial, there is an isomorphism $\Psi : E \to F \otimes \mathcal{V}$. The inclusion map $\iota : \mathcal{V} \to \mathbb{C}T\mathcal{M}$ gives the dual map

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Get a homomorphism

$\mathcal{V} \otimes \mathcal{V}^*$ is canonically trivial

$$E \otimes \mathbb{C}T^*\mathcal{M} \xrightarrow{\Psi\otimes\iota^*} F \otimes \mathcal{V} \otimes \mathcal{V}^* = F$$

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> *If $\dim \mathcal{M} \geq 3$ and E is not isomorphic to F, no pseudo-differential $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ can be elliptic.[6]*

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If P is globally C^∞-hypoelliptic, then for every $r, s \in \mathbb{R}$ there is $t \in \mathbb{R}$ and $C > 0$ such that[7]

$$\|v\|_r \leq C\big(\|Pv\|_t + \|v\|_s\big) \quad \forall v \in C^\infty(\mathcal{M}; E).$$

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> *There are smooth functions $h_1, \dots, h_\ell$ such that*
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> $$\forall f \in C^\infty(\mathcal{M}; E^*)\ \exists w \in C^{-\infty}(\mathcal{M}; F^*),\ c_j \in \mathbb{C}\ s.t.\ P^\dagger w = f + \sum c_j h_j.$$

Solvability modulo a space of finite codimension

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What is the Hamiltonian vector field of $\boldsymbol{\sigma}(P)$?

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The section $p : \mathcal{Y} \to G$ determines a section of $\pi^* G \to SG$:

$$SG \ni \eta \longmapsto \pi^* p(\eta) = (\eta, \wp(\pi(\eta))) = \check{p}(\eta)\lambda(\eta)$$

where now $\check{p} : SG \to \mathbb{C}$.

Fix a Hermitian connection on G and use it to construct a horizontal line bundle $\mathcal{H} \subset TG$:

For all ϕ, $d\rho : \mathcal{H}_\phi \to T_{\rho(\phi)}\mathcal{Y}$ is an isomorphism.

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Let $\mathcal{T}$ be the infinitesimal generator of the S^1-action (multiplication by e^{it}) on SG. Let θ be the 1-form on SG which vanishes on $\mathcal{H}|_{SE}$ and has $\langle \theta, \mathcal{T} \rangle = 1$. ($\theta =$ the connection form)

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Local expression of $\check{H}_{\check{p}}$

Let $x^1, \dots, x^n, \xi_1, \dots, \xi_n$ be local canonical coordinates near $\nu_0 \in T^*\mathcal{M}\backslash 0$. With a trivialization of G near ν_0 they give coordinates on SG with which:

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The vector fields

$$X_j = \partial_{x^j} - \theta_j \partial_t, \quad \Xi^j = \partial_{\xi_j} - \theta^j \partial_t, \quad j = 1, \dots, n$$

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give a frame of the subbundle $\mathcal{H}$, the kernel of θ, over V. Together with $\mathcal{T} = \partial_t$ they give a frame of the tangent bundle of SG over V. The dual frame is

$$dx^1, \dots, dx^n, d\xi_1, \dots, d\xi_n, \theta.$$

The functions x^i, ξ_j really mean $\pi^* x^i$, $\pi^* \xi_j$.

So if $f : SG \to \mathbb{R}$ is smooth, then

$$df = \mathcal{T}f\, \theta + \sum_j X_j f\, dx^j + \Xi^j f\, d\xi_j.$$

Thus the form on $T^*_{\pi(\eta)}\mathcal{M}$ that corresponds to $df(\eta)$ is

$$\sum_j X_j f(\eta)\, dx^j + \Xi^j f(\eta)\, d\xi_j.$$

The vector v corresponding to it by the symplectic structure of $T^*\mathcal{M}$ is

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Note that $\hat{H}_f f = 0$.

If $f = \pi^* g$, then $\hat{H}_f$ is just the horizontal lift of H_g.

By construction, the vector field is horizontal (lies in $\mathcal{H}$). Suppose P is of principal type and Being of principal type is a local condition!

$$\|v\|_{s+1} \leq C\big(\|Pv\|_t + \|v\|_s\big) \quad \forall v \in C^\infty(\mathcal{M}; E).$$

holds for some s.

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If $\check{\gamma} : I \to SG$ is an integral curve of $\check{H}_{\Re\check{p}}$ in $\{\Re\check{p} = 0\}$ whose projection on $T^*\mathcal{M}\backslash 0$ has no self-intersection, then $\Im\check{p} \circ \gamma$ does not change sign.

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The proof is as that of the necessity[1] of Condition (ψ) for scalar operators, observing that there is a canonical transformation[2] mapping the projection on $T^*\mathcal{M}\backslash 0$ of the curve to $t \mapsto (0, t; \xi_0', 0)$ and that complex line bundles are trivial along curves.

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In somewhat more detail:

Suppose that $\check{\gamma} : I \to SG$ is an integral curve of $\check{H}_{\Re p}$ in $\{\Re p = 0\}$ whose projection on $T^*\mathcal{M}\backslash 0$ has no self-intersection; we can assume I is a closed bounded interval. There is a homogeneous canonical transformation Ψ from a conic neighborhood of $K = (\pi \circ \check{\gamma})(I)$ to a conic neighborhood of

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If P is analytic, then actual changes of sign can only occur at isolated points. In this case microlocal solvability for scalar operators (applied to the transpose $P^\dagger$) gives the statement.

Back to first order differential operators

Assume $\mathcal{M}$, E, F and $P \in \mathrm{Diff}^1(\mathcal{M}; E, F)$ are real-analytic, P is C^∞-hypoelliptic and $d_\xi \boldsymbol{\sigma}(P) \neq 0$.

$\mathcal{M}$ a closed orientable surface.

The following comes from joint work with H. Jacobowitz.[3]

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Construct the fiberwise projectivization of $\mathbb{C}T\mathcal{M}$:

$$\begin{tikzcd} & \mathbb{P}\mathbb{C}T\mathcal{M} \arrow[dl] & S^2 \arrow[l, hook'] \\ \mathcal{M} & & \end{tikzcd}$$

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$\mathcal{V}$ satisfies $(\mathscr{P})$ iff $[\mathcal{V}]$ is contained in the closure of one of the components of $\mathbb{P}\mathbb{C}T\mathcal{M} \backslash \mathbb{P}T\mathcal{M}$.

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As a special case/example, suppose $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is a line subbundle ($\mathcal{M}$ and $\mathcal{V}$ real-analytic). It defines the (very short) complex

$$0 \to C^\infty(\mathcal{M}) \xrightarrow{\mathbb{D}} C^\infty(\mathcal{M}; \mathcal{V}^*) \to 0.$$

So E is the trivial line bundle and $F = \mathcal{V}^*$ and of course

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