

On global hypoellipticity

Gerardo A. Mendoza

Temple University

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Suppose \mathcal{M} is closed, two-dimensional, orientable. If L is a globally defined nowhere vanishing complex vector field on \mathcal{M} which is globally hypoelliptic, then \mathcal{M} is a torus.

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Functional-analytic properties of the partial differential operator have implications on the nature of the topological objects involved (the manifold or the vector bundles).

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Suppose \mathcal{M} is a real-analytic orientable surface without boundary, $E, F \rightarrow \mathcal{M}$ are real-analytic complex line bundles, and $P : C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F)$ is a first order real-analytic differential operator of principal type.

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Euler characteristic of \mathcal{M}

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If P is globally C^∞ -hypoelliptic, then $Pu \in C^\infty(\mathcal{M}) \implies u \in C^\infty(\mathcal{M})$

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Principal type is meant in the more restricted sense that $d_\xi \sigma(P)$ vanishes nowhere on $T^*\mathcal{M} \setminus 0$.

$\sigma(P)$ is linear in the fiber variable of $T^*\mathcal{M}$;
the condition is $\sigma(P)|_{\text{Fiber}} \neq 0$

The existence of a first order differential operator $P : C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F)$ of principal type implies the existence of a line bundle \mathcal{V} such that $F \otimes \mathcal{V} \otimes E^$ is trivial.*

The existence of a first order differential operator $P : C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F)$ of principal type is equivalent to the existence of a line bundle \mathcal{V} such that $F \otimes \mathcal{V} \otimes E^*$ is trivial.

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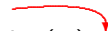
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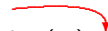
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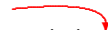
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Setting $E = F =$ trivial line bundle

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Actually, hypoellipticity + principal type + $E = F$ already implies $\mathcal{M} = \text{torus}$.
(Hypoellipticity is needed to get $\deg(\mathcal{V}) = \pm\chi(\mathcal{M})$)

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The relevant condition is hypoellipticity, not the existence of a differential operator of a certain kind

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In the context of ellipticity...

In joint work with H. Jacobowitz⁴ we showed among other things:

- If \mathcal{M} is a smooth orientable manifold and $E, F \rightarrow \mathcal{M}$ are smooth complex vector bundles of the same rank, then the existence of an elliptic pseudo-differential operator

$$P : C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F) \quad (\dagger)$$

implies that

$$c(F) - c(E) = k \mathbf{e}_{\mathcal{M}} \quad (\ddagger)$$

for some $k \in \mathbb{Z}$.

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$e_{\mathcal{M}}$ is the Euler class of \mathcal{M} : $\int_{\mathcal{M}} e_{\mathcal{M}} = \chi(\mathcal{M})$.

Also

- If \mathcal{M} is compact without boundary, $\dim \mathcal{M} = 2$ and $\chi(\mathcal{M}) \neq 0$, and if (\dagger) is an elliptic pseudo-differential operator and (\ddagger) holds, then $\sigma(P)$ is homotopic through elliptic symbols, to that of an elliptic differential operator of order $|k|$.

Why $\neq 0$? 

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I'll show:

- If \mathcal{M} is a smooth orientable manifold and $E, F \rightarrow \mathcal{M}$ are smooth complex vector bundles of the same rank, then the existence of an elliptic pseudo-differential operator

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implies that

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Proof:

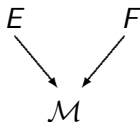
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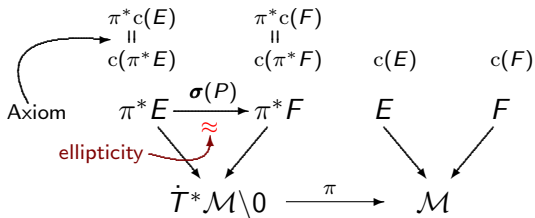
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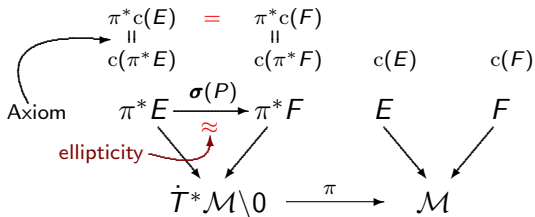
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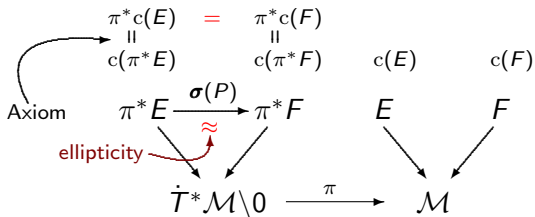


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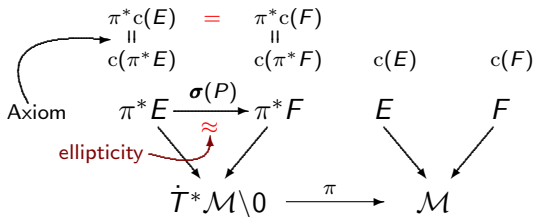
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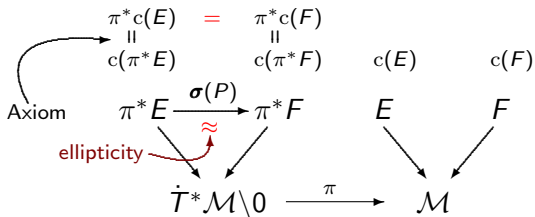


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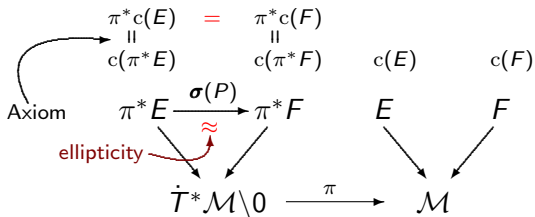
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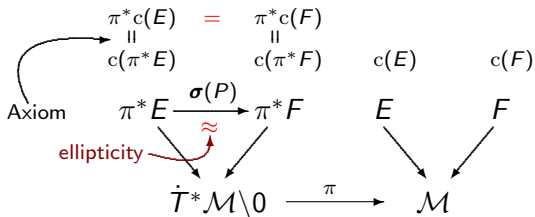
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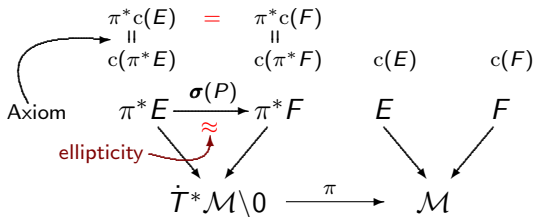
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First order differential operators of principal type

There is $P \in \text{Diff}^1(\mathcal{M}; E, F)$ of principal type iff there is a line subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ such that $F \otimes \mathcal{V} \otimes E^$ is trivial.*

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$\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is the annihilator of $\{\xi \in \mathbb{C}T^*\mathcal{M} : \sigma(P)(\xi) = 0\}$,
but we need to see $F \otimes \mathcal{V} \otimes E^*$
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With local frames ϕ and ψ of E, F over $U \subset \mathcal{M}$:
 E, F are line bundles, $\dim \mathcal{M}$ arbitrary
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$$P(h\phi) = (Lh + ch)\psi$$

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$$\deg E - \deg F = \deg \mathcal{V}.$$

$$0 = c_1(F \otimes \mathcal{V} \otimes E^*) = c_1(F) + c_1(\mathcal{V}) - c_1(E)$$

$$\deg E = \int_M c_1(E)$$

Since $F \otimes \mathcal{V} \otimes E^*$ is trivial, there is an isomorphism $\Psi : E \rightarrow F \otimes \mathcal{V}$. The inclusion map $\iota : \mathcal{V} \rightarrow \mathbb{C}T\mathcal{M}$ gives the dual map

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Let \mathcal{M} be closed, orientable, of arbitrary dimension n . Any vector bundle $\mathcal{V} \rightarrow \mathcal{M}$ of rank $r \leq (n + 1)/2$ can be realized as a subbundle of $\mathbb{C}T\mathcal{M}$.

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Consequently, given arbitrary complex line bundles $E, F \rightarrow \mathcal{M}$, there is a line subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ such that $\mathcal{V} \approx F^* \otimes E$.

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A well known fact in the theory of fiber bundles is:

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If $\dim \mathcal{M} \geq 3$ and E is not isomorphic to F , no pseudo-differential $P : C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F)$ can be elliptic.⁶

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$$\|v\|_r \leq C(\|Pv\|_t + \|v\|_s) \quad \forall v \in C^\infty(\mathcal{M}; E).$$

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There are smooth functions h_1, \dots, h_ℓ such that

$$\forall f \in C^\infty(\mathcal{M}; E^*) \exists w \in C^{-\infty}(\mathcal{M}; F^*), c_j \in \mathbb{C} \text{ s.t. } P^\dagger w = f + \sum c_j h_j.$$

Solvability modulo a space of finite codimension

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When P is a scalar operator of principal type, solvability of P^\dagger is equivalent to Condition (\mathcal{P}) of Nirenberg-Treves:^{9,10}

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What is the Hamiltonian vector field of $\sigma(P)$?

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View the symbol as a section of $G = \pi^*(F \otimes E^*) \approx \text{Hom}(\pi^*E, \pi^*F)$, another line bundle. Write \mathcal{Y} for the manifold $T^*\mathcal{M} \setminus 0$:

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where now $\check{\rho} : SG \rightarrow \mathbb{C}$.

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Local expression of $\check{H}_{\check{\rho}}$

Let $x^1, \dots, x^n, \xi_1, \dots, \xi_n$ be local canonical coordinates near $\nu_0 \in T^*\mathcal{M} \setminus 0$.
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$$X_j = \partial_{x^j} - \theta_j \partial_t, \quad \Xi^j = \partial_{\xi_j} - \theta^j \partial_t, \quad j = 1, \dots, n$$

give a frame of the subbundle \mathcal{H} , the kernel of θ , over V .

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give a frame of the subbundle \mathcal{H} , the kernel of θ , over V . Together with $\mathcal{T} = \partial_t$ they give a frame of the tangent bundle of SG over V . The dual frame is

$$dx^1, \dots, dx^n, d\xi_1, \dots, d\xi_n, \theta.$$

The functions x^i, ξ_j really mean $\pi^*x^i, \pi^*\xi_j$.

So if $f : SG \rightarrow \mathbb{R}$ is smooth, then

$$df = \mathcal{T}f \theta + \sum_j X_j f dx^j + \Xi^j f d\xi_j.$$

Thus the form on $T_{\pi(\eta)}^* \mathcal{M}$ that corresponds to $df(\eta)$ is

$$\sum_j X_j f(\eta) dx^j + \Xi^j f(\eta) d\xi_j.$$

The vector v corresponding to it by the symplectic structure of $T^* \mathcal{M}$ is

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Note that $\hat{H}_f f = 0$.

If $f = \pi^* g$, then \hat{H}_f is just the horizontal lift of H_g .

By construction, the vector field is horizontal (lies in \mathcal{H}). Suppose P is of principal type and

Being of principal type is a local condition!

$$\|v\|_{s+1} \leq C(\|Pv\|_t + \|v\|_s) \quad \forall v \in C^\infty(\mathcal{M}; E).$$

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If $\check{\gamma} : I \rightarrow SG$ is an integral curve of $\check{H}_{\mathfrak{R}\check{p}}$ in $\{\mathfrak{R}\check{p} = 0\}$ whose projection on $T^\mathcal{M} \setminus 0$ has no self-intersection, then $\mathfrak{S}\check{p} \circ \gamma$ does not change sign.*

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The proof is as that of the necessity¹ of Condition (ψ) for scalar operators, observing that there is a canonical transformation² mapping the projection on $T^*\mathcal{M} \setminus 0$ of the curve to $t \mapsto (0, t; \xi'_0, 0)$ and that complex line bundles are trivial along curves.

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In somewhat more detail:

Suppose that $\check{\gamma} : I \rightarrow SG$ is an integral curve of $\check{H}_{\Re p}$ in $\{\Re p = 0\}$ whose projection on $T^*\mathcal{M} \setminus 0$ has no self-intersection; we can assume I is a closed bounded interval. There is a homogeneous canonical transformation Ψ from a conic neighborhood of $K = (\pi \circ \check{\gamma})(I)$ to a conic neighborhood of

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If P is analytic, then actual changes of sign can only occur at isolated points. In this case microlocal solvability for scalar operators (applied to the transpose P^\dagger) gives the statement.

Back to first order differential operators

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\mathcal{M} a closed orientable surface.

The following comes from joint work with H. Jacobowitz.³

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Construct the fiberwise projectivization of $\mathbb{C}T\mathcal{M}$:

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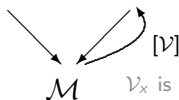
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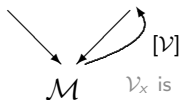
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\mathcal{V} satisfies (\mathcal{P}) iff $[\mathcal{V}]$ is contained in the closure of one of the components of $\mathbb{P}\mathbb{C}T\mathcal{M} \setminus \mathbb{P}T\mathcal{M}$.

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As a special case/example, suppose $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is a line subbundle (\mathcal{M} and \mathcal{V} real-analytic). It defines the (very short) complex

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So E is the trivial line bundle and $F = \mathcal{V}^*$ and of course

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