Embedding theorems of manifolds with $\mathbb{R}$-action

Gerardo Mendoza

Temple University

Serra Negra, August 2011

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Both proofs use an idea of S. Bochner, *Analytic mapping of compact Riemann spaces into Euclidean space*, Duke Math. J. **3** (1937), no. 2, 339354.

The proof of the second result also uses ideas from L. Boutet de Monvel in *Intégration des équations de Cauchy-Riemann induites formelles*, Séminaire Goulaouic-Lions-Schwartz 1974–1975, Exp. No. 9.

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Let $\mathscr{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ where

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Let $\boldsymbol{\beta}$ be the class of β.

Let $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with $\dim \mathcal{N} \geq 5$. Fix $\boldsymbol{\beta}$. The following are equivalent:

- *$\exists \beta \in \boldsymbol{\beta}$ such that $\overline{\mathcal{K}}_\beta$ is definite.*
- *$\exists \beta \in \boldsymbol{\beta}$ and an equivariant CR embedding $F : \mathcal{N}, \overline{\mathcal{K}}_\beta \to S^{2N-1}$ for some N, with $F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j})$ and all τ_j of the same sign.*

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$$\{\phi_{\tau,\lambda,j}\in\mathcal{E}_{\tau,\lambda} : j=1,\dots,\dim\mathcal{E}_{\tau,\lambda}\},\quad (\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$$

(1) for all $p_0\in\mathcal{N}$, $\mathbb{C}T^*_{p_0}\mathcal{N} = \operatorname{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau,\lambda)\in \operatorname{spec}(-i\mathcal{T},\Delta),\ j=1,\dots,N_{\tau,\lambda}\}$;

(2) the functions $\phi_{\tau,\lambda,j}$, $(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$, $j=1,\dots,N_{\tau,\lambda}$, *separate points of* $\mathcal{N}$.

If $\tau_\ell<0$, replace $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ by $\overline{\phi}_{\tau_\ell,\lambda_\ell,j_\ell}$.

Continuity, compactness and (1) give an immersion

$$F=(\phi_{\tau_1,\lambda_1,j_1},\dots,\phi_{\tau_N,\lambda_N,j_N}) : SE\to\mathbb{C}^N\backslash 0$$

Use property (2) to ensure that the functions $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ separate points.

Outline of proofs

Embedding in $\mathbb{C}^N \backslash 0$ of $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$.

Pick $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$ and a $\mathcal{T}$-invariant metric, let Δ be the Laplacian. Note that Δ and $\mathcal{T}$ commute. Let

$\mathcal{E}_\lambda = \ker(\Delta - \lambda)$ is invariant under $\mathcal{T}$, $-i\mathcal{T}$ is symmetric, so $\operatorname{spec}(-iT|_{\mathcal{E}_\lambda}) \subset \mathbb{R}$

$$\mathcal{E}_{\tau,\lambda} = \{\phi \in C^\infty(\mathcal{N}) : \Delta\phi = \lambda\phi, -i\mathcal{T}\phi = \tau\phi\},$$
$$\operatorname{spec}(-i\mathcal{T}, \Delta) = \{(\tau, \lambda) : \mathcal{E}_{\tau,\lambda} \neq 0\}.$$

$\mathfrak{a}_t$ = one parameter group of diffeos generated by $\mathcal{T}$

$-i\mathcal{T}\phi = \tau\phi \implies$ $\boxed{\phi(\mathfrak{a}_t p) = e^{i\tau t}\phi(p)}$

Pick orthonormal bases

$$\{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \dots, \dim \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \operatorname{spec}(-i\mathcal{T}, \Delta)$$

(1) for all $p_0 \in \mathcal{N}$, $\mathbb{C}T^*_{p_0}\mathcal{N} = \operatorname{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \operatorname{spec}(-i\mathcal{T}, \Delta),\ j = 1, \dots, N_{\tau,\lambda}\}$;

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If $\tau_\ell < 0$, replace $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ by $\overline{\phi}_{\tau_\ell,\lambda_\ell,j_\ell}$.

Continuity, compactness and (1) give an immersion

$$F = (\phi_{\tau_1,\lambda_1,j_1}, \dots, \phi_{\tau_N,\lambda_N,j_N}) : SE \to \mathbb{C}^N \backslash 0$$

Use property (2) to ensure that the functions $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ separate points.

Outline of proofs

Pick $(\mathcal{N},\mathcal{T})\in\mathscr{F}$ and a $\mathcal{T}$-invariant metric, let Δ be the Laplacian. Note that Δ and $\mathcal{T}$ commute. Let

$\mathcal{E}_\lambda=\ker(\Delta-\lambda)$ is invariant under $\mathcal{T}$, $-i\mathcal{T}$ is symmetric, so $\operatorname{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda})\subset\mathbb{R}$

$$\mathcal{E}_{\tau,\lambda}=\{\phi\in C^\infty(\mathcal{N}):\Delta\phi=\lambda\phi,\ \boxed{-i\mathcal{T}\phi=\tau\phi}\},$$
$$\operatorname{spec}(-i\mathcal{T},\Delta)=\{(\tau,\lambda):\mathcal{E}_{\tau,\lambda}\neq 0\}.$$

$\mathfrak{a}_t$ = one parameter group of diffeos generated by $\mathcal{T}$

$-i\mathcal{T}\phi=\tau\phi\implies$ $\boxed{\phi(\mathfrak{a}_t p)=e^{i\tau t}\phi(p)}$

Pick orthonormal bases

$$\{\phi_{\tau,\lambda,j}\in\mathcal{E}_{\tau,\lambda}:j=1,\dots,\dim\mathcal{E}_{\tau,\lambda}\},\quad (\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$$

(1) for all $p_0\in\mathcal{N}$, $\mathbb{C}T^*_{p_0}\mathcal{N}=\operatorname{span}\{d\phi_{\tau,\lambda,j}(p_0):(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta),\ j=1,\dots,N_{\tau,\lambda}\}$;

(2) the functions $\phi_{\tau,\lambda,j}$, $(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$, $j=1,\dots,N_{\tau,\lambda}$, separate points of $\mathcal{N}$.

If $\tau_\ell<0$, replace $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ by $\overline{\phi}_{\tau_\ell,\lambda_\ell,j_\ell}$.

Continuity, compactness and (1) give an immersion

$$F=(\phi_{\tau_1,\lambda_1,j_1},\dots,\phi_{\tau_N,\lambda_N,j_N}):SE\to\mathbb{C}^N\backslash 0$$

Use property (2) to ensure that the functions $\phi_{\tau_\ell,\lambda_\ell,j_\ell}$ separate points. Since $\phi_{\tau_\ell,\lambda_\ell,j_\ell}(\mathfrak{a}_t p)=e^{i\tau_\ell t}\phi_{\tau_\ell,\lambda_\ell,j_\ell}(p)$, $F_*\mathcal{T}=i\sum\tau_\ell(z^\ell\partial_{z^\ell}-\overline{z}^\ell\partial_{\overline{z}^\ell})$. ☐

Outline of proofs

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CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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$\mathscr{F}_{\mathrm{ell}} =$ set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ s.t.:
- $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$;
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- $\exists \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{\mathbb{D}}\beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta \sim \beta'$ iff there is u real-valued such that

$$\beta' - \beta = \overline{\mathbb{D}}u.$$

Outline of proofs

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Suppose $\beta \in \boldsymbol{\beta}$, let $\overline{\mathcal{K}}_\beta = \ker \beta$

From Cartan's formula

$$\mathcal{L}_{\mathcal{T}}\beta = \overline{\mathbb{D}}(\mathbf{i}_{\mathcal{T}}\beta) + \mathbf{i}_{\mathcal{T}}\overline{\mathbb{D}}\beta = 0$$

deduce $\mathfrak{a}_t : \overline{\mathcal{K}}_\beta \to \overline{\mathcal{K}}_\beta$.

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CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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Let $\mathcal{H}_\beta$ be the subbundle of $T\mathcal{N}$ s.t.

$$\mathbb{C}\mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta$$

Let $J{:}\mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_\beta = \{v + iJv : v \in \mathcal{H}_\beta\}$

Pick a $\mathcal{T}$-invariant metric g. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \tfrac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $\mathcal{T}$-invariant hermitian metric on $\mathcal{H}_\beta$.

CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathscr{F}_{\rm ell}$ with definite $\boldsymbol{\beta}$.

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Outline of proofs

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Outline of proofs

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With this data (hermitian metric, Riemannian measure), let

$$\Box_b = \text{Laplacian of } \overline{\partial}_b \text{ complex in any degree.}$$

Outline of proofs

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Note $\mathcal{L}_\mathcal{T}\Box_b = \Box_b\mathcal{L}_\mathcal{T}$.

Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

Let $\mathscr{H}^q_{\overline{\partial}_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \overline{\mathcal{K}}^*)$.

Let $\mathcal{D} = \{\phi \in \mathscr{H}^q_{\overline{\partial}_b} : \mathcal{L}_{\mathcal{T}}\phi \in L^2\}$

$$-i\mathcal{L}_{\mathcal{T}} : \mathcal{D} \subset \mathscr{H}^q_{\overline{\partial}_b} \to \mathscr{H}^q_{\overline{\partial}_b}$$

is a selfadjoint operator with compact parametrix.

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$$\operatorname{spec}^q_0(-i\mathcal{L}_{\mathcal{T}}) = \textit{spectrum of } (*)$$

is a discrete subset of $\mathbb{R}$.

Outline of proofs

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The proof exploits $[\Box_b,\mathcal{L}_\mathcal{T}]=0$ plus the fact that $\Box_b-\mathcal{L}^2_\mathcal{T}$ is elliptic (and that $\mathcal{N}$ is compact).

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Theorem. *Suppose that* Levi_θ *is nondegenerate with* k *positive and* $n-k$ *negative eigenvalues. Then*

$\dim \mathcal{N} = 2n+1$

$\mathrm{spec}^q_0(-i\mathcal{L}_\mathcal{T})$ *is finite if* $q \neq k,\ n-k$;
$\mathrm{spec}^k_0(-i\mathcal{L}_\mathcal{T})$ *contains only finitely many positive elements, and*
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Outline of proofs

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The case $k=n$ (or $k=0$) is like Kodaira's theorem on vanishing of cohomology.

Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

Let $\mathscr{H}^q_{\overline{\partial}_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \overline{\mathcal{K}}^*)$.

Let $\mathcal{D} = \{\phi \in \mathscr{H}^q_{\overline{\partial}_b} : \mathcal{L}_{\mathcal{T}}\phi \in L^2\}$

$$(*) \qquad -i\mathcal{L}_{\mathcal{T}} : \mathcal{D} \subset \mathscr{H}^q_{\overline{\partial}_b} \to \mathscr{H}^q_{\overline{\partial}_b}$$

is a selfadjoint operator with compact parametrix. Consequently

$$\mathrm{spec}^q_0(-i\mathcal{L}_{\mathcal{T}}) = \textit{spectrum of } (*)$$

is a discrete subset of $\mathbb{R}$.

The proof exploits $[\Box_b, \mathcal{L}_{\mathcal{T}}] = 0$ plus the fact that $\Box_b - \mathcal{L}^2_{\mathcal{T}}$ is elliptic (and that $\mathcal{N}$ is compact).

Theorem. *Suppose that* Levi_θ *is nondegenerate with* k *positive and* $n - k$ *negative eigenvalues. Then*

$\dim \mathcal{N} = 2n + 1$

$\mathrm{spec}^q_0(-i\mathcal{L}_{\mathcal{T}})$ *is finite if* $q \neq k,\ n - k$;
$\mathrm{spec}^k_0(-i\mathcal{L}_{\mathcal{T}})$ *contains only finitely many positive elements, and*
$\mathrm{spec}^{n-k}_0(-i\mathcal{L}_{\mathcal{T}})$ *contains only finitely many negative elements.*

The case $k = n$ (or $k = 0$) is like Kodaira's theorem on vanishing of cohomology.

If $k = n$ *(or* 0*), then* $\mathrm{spec}^0_0(i\mathcal{L}_{\mathcal{T}})$ *contains no negative (positive) elements.*

Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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$\operatorname{spec}^q_0(-i\mathcal{L}_\mathcal{T})$ *is finite if* $q\neq k,\ n-k$;
$\operatorname{spec}^k_0(-i\mathcal{L}_\mathcal{T})$ *contains only finitely many positive elements, and*
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This is because $\operatorname{spec}^0_0(i\mathcal{L}_\mathcal{T})$ is an additive subgroup of $\mathbb{R}$.

The case $k=n$ (or $k=0$) is like Kodaira's theorem on vanishing of cohomology.

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Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

Let $\{\phi_\ell\}_{\ell=0}^{\infty}$ *be an orthonormal basis of* $\mathscr{H}^0_{\overline{\partial}_b}$ *consisting of eigenvectors of* $-i\mathcal{L}_{\mathcal{T}}$, $\phi_\ell \in \mathscr{H}^q_{\overline{\partial}_b, \tau_\ell}$. *Then there are* $C, \mu > 0$ *such that*

$$\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^{\mu} \text{ for all } p \in \mathcal{N},\ \ell \in \mathbb{N}_0.$$

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This implies: The Fourier series of $u \in C^\infty(\mathcal{N}) \cap \mathscr{H}^0_{\overline{\partial}_b}$ converges in C^∞.

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Consequence:

(1) for all $p_0\in\mathcal{N}$, $\operatorname{span}\{d\phi_\ell(p_0):\ell=0,1,\dots\}$ is the annihilator of $\overline{\mathcal{K}}_\beta$ in $\mathbb{C}T^*_{p_0}\mathcal{N}$;

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Theorem. *Suppose* $\overline{\mathcal{K}}_\beta$ *is definite. There is an embedding* $F:\mathcal{N}\to\mathbb{C}^N\backslash 0$ *such that*

$$F_*\mathcal{T}=i\textstyle\sum_j\tau_j(z^j\partial_{z^j}-\overline{z}^j\partial_{\overline{z}^j})$$

and all τ_j *of the same sign.*

The proofs of (1) and (2) use ideas of Boutet de Monvel

F is constructed using eigenfunctions.

Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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This is an embedding into $\mathbb{C}^N \backslash 0$. Getting one into S^{2N-1} requires changing β within the class $\boldsymbol{\beta}$.

F is constructed using eigenfunctions.

Outline of proofs

CR embedding in S^{2N-1} of $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathscr{F}_{\mathrm{ell}}$ with definite $\boldsymbol{\beta}$.

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This is an embedding into $\mathbb{C}^N\backslash 0$. Getting one into S^{2N-1} requires changing β within the class $\boldsymbol{\beta}$.

This is like changing the Hermitian metric but not the holomorphic structure of a line bundle.

F is constructed using eigenfunctions.

End