Geometric aspects of the spectrum of extensions of elliptic cone operators

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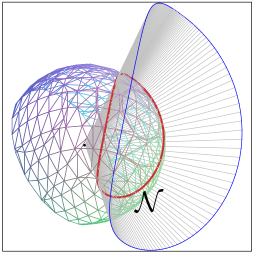
Outline

Manifolds with conical singularities

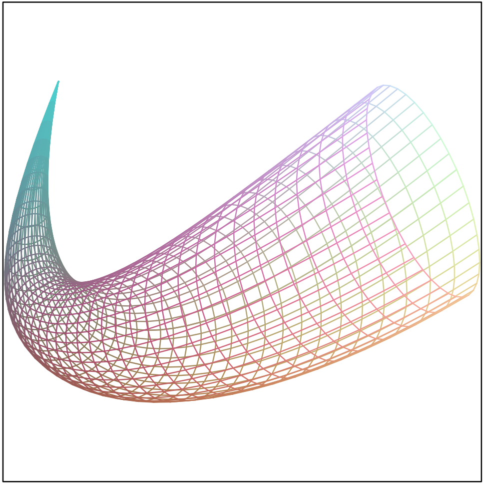
A local model for a manifold with conical singularities is a set of the form

$$\mathcal{C}_{\mathcal{N}} = \{\mathfrak{r}p \in \mathbb{R}^m : \mathfrak{r} \geq 0,\ p \in \mathcal{N}\}$$

where $\mathcal{N}$ is a smooth submanifold of S^{m-1}. The set $\mathcal{C}_{\mathcal{N}} \backslash \{0\}$ is a smooth manifold of dimension $\dim \mathcal{N} + 1$.



More generally we may talk about manifolds with conical singularities by defining first the notion of local diffeomorphism between model cones.



Spherical blowup

The spherical blowup of a model cone $\mathcal{C}_\mathcal{N} \subset \mathbb{R}^m$ is the manifold $[0, \infty) \times \mathcal{N}$ together with the map

$$\wp : [0, \infty) \times \mathcal{N} \to \mathcal{C}_\mathcal{N}, \quad \wp(\mathfrak{r}, p) = \mathfrak{r}p$$

The blowup is a manifold with boundary ($= \{0\} \times \mathcal{N} \approx \mathcal{N}$). This is just putting polar coordinates on the model cone.

One also defines the spherical blowup of a manifold with conical singularities.

If $\mathcal{M}_0$ is a manifold with conical singularities, then there is a discrete (closed) subset $\mathscr{S} \subset \mathcal{M}_0$ such that

- each $p_0 \in \mathscr{S}$ has a neighborhood diffeomorphic to a neighborhood of the apex of a model cone, and
- $\mathcal{M}_0 \backslash \mathscr{S}$ is a smooth manifold.

The blowup, $\mathcal{M}$, is a manifold with boundary. Each component of $\partial \mathcal{M}$ is the base of a model cone. There is a "smooth" map

$$\wp : \mathcal{M} \to \mathcal{M}_0, \quad \mathring{\mathcal{M}} \xrightarrow{\cong} \mathcal{M}_0 \backslash \mathscr{S}, \quad \wp(\partial \mathcal{M}) = \mathscr{S}$$

Analysis done on $\mathcal{M}$ rather than on $\mathcal{M}_0$.

Henceforth

- $\mathcal{M}$ is a smooth $(n+1)$-manifold with boundary and
- $\mathfrak{r} : \mathcal{M} \to \mathbb{R}$ a smooth defining function of $\partial\mathcal{M}$ with $\mathfrak{r} > 0$ in $\mathring{\mathcal{M}}$.

What is a metric?

Metrics

Motivation: $\mathbb{R}^m$ is itself a model cone with base S^{m-1}.

The spherical blowup of $\mathbb{R}^m$ at 0 is $\mathcal{M} = [0,\infty) \times S^{m-1}$ with blowdown map

$$\wp : [0,\infty) \times S^{m-1} \to \mathbb{R}^m,\ \ \wp(\mathfrak{r},\omega) = \mathfrak{r}\omega,$$
$$\omega = (\omega^1, \dots, \omega^m) \in S^{m-1}.$$

The standard metric on $\mathbb{R}^m$ lifts to

$$\wp^* \sum dx^j \otimes dx^j = \sum d(\mathfrak{r}\omega^j) \otimes d(\mathfrak{r}\omega^j) = d\mathfrak{r} \otimes d\mathfrak{r} + \sum \mathfrak{r} d\omega^j \otimes \mathfrak{r} d\omega^j$$

Built out of $d\mathfrak{r}$ and $\mathfrak{r}d\omega^j$'s

Definition (Cone metric)

A cone metric on $\mathcal{M}$ is a smooth symmetric 2-tensor g on $\mathcal{M}$ which is

- positive definite in $\mathring{\mathcal{M}}$ and
- near any $p_0 \in \partial\mathcal{M}$, in coordinates $\mathfrak{r}, x^1, \dots, x^n$, g has the form

$$g_{0,0}\, d\mathfrak{r} \otimes d\mathfrak{r} + \sum_{j=1}^{n} g_{0,j}(d\mathfrak{r} \otimes \mathfrak{r}dx^j + \mathfrak{r}dx^j \otimes d\mathfrak{r}) + \sum_{i,j=1}^{n} g_{i,j}\, \mathfrak{r}dx^i \otimes \mathfrak{r}dx^j$$

with $[g_{\mu,\nu}]_{\mu,\nu=0}^{n}$ smooth positive definite.

Vector fields of the form

$$X = a_0 \partial_{\mathfrak{r}} + \sum_{j=1}^{n} a_j \frac{1}{\mathfrak{r}} \partial_{x_j}, \quad a_\nu \text{ smooth up to } \partial\mathcal{M}$$

have bounded pointwise norm. The Laplacian of a cone metric is made up of such vector fields with smooth coefficients.

Definition (b-metric)

A b-metric on $\mathcal{M}$ is a symmetric 2-tensor g_b on $\mathring{\mathcal{M}}$ of the form $\mathfrak{r}^{-2}g$ where g is a cone metric.

So, locally

$$g_b = g_{0,0}\frac{d\mathfrak{r}}{\mathfrak{r}} \otimes \frac{d\mathfrak{r}}{\mathfrak{r}} + \sum_j g_{0,j}(\frac{d\mathfrak{r}}{\mathfrak{r}} \otimes dx^j + dx^j \otimes \frac{d\mathfrak{r}}{\mathfrak{r}}) + \sum_{i,j} g_{i,j}\, dx^i \otimes dx^j$$

Vector fields of the form

$$X = a_0 \mathfrak{r}\partial_{\mathfrak{r}} + \sum_j a_j \partial_{x_j}$$

have bounded pointwise norm. Again the Laplacian of a b-metric is made up of such vector fields with smooth coefficients.

L^2 spaces

A b-density on $\mathcal{M}$ is a density $\mathfrak{m}_b$ of the form $\mathfrak{r}^{-1}\mathfrak{m}$ where $\mathfrak{m}$ is a smooth positive density.

The Riemannian density of a b-metric is a b-density. The L^2 spaces with respect to $\mathfrak{m}_b$ are denoted $L^2_b(\mathcal{M}; E)$.

The Riemannian density of a cone metric has the form $\mathfrak{m}_c = \mathfrak{r}^{n+1}\mathfrak{m}_b$ for some $\mathfrak{m}_b$.

If $u \in L^2(\mathcal{M}, \mathfrak{m}_c)$, then

$$\int |u|^2 \mathfrak{r}^{n+1}\mathfrak{m}_b = \int |\mathfrak{r}^{(n+1)/2} u|^2 \mathfrak{m}_b$$

is finite, so $\mathfrak{r}^{(n+1)/2} u \in L^2_b$. This fact is expressed

$$u \in \mathfrak{r}^{-(n+1)/2} L^2_b.$$

The b-tangent bundle[1]

Let $C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$ be the space of smooth vector fields on $\mathcal{M}$ that are tangent to $\partial\mathcal{M}$.
There is a vector bundle ${}^bT\mathcal{M} \to \mathcal{M}$ and bundle map $\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$ such that

$$\mathrm{ev}_* : C^\infty(\mathcal{M}; {}^bT\mathcal{M}) \to C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$$

is a $C^\infty(\mathcal{M})$-module isomorphism.
A b-metric can be viewed as a smooth Riemannian metric on ${}^bT\mathcal{M}$, a symmetric section of

$${}^bT^*\mathcal{M} \otimes {}^bT^*\mathcal{M}.$$

The space of linear differential operators $C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ of order m with smooth coefficients based on sections of ${}^bT\mathcal{M}$ is denoted

$$\mathrm{Diff}^m_b(\mathcal{M}; E)$$

[1] R. B. Melrose, *Transformation of boundary problems* Acta Math. **147** (1981), 149–236.

The Laplacian of a b-metric is an element of $\operatorname{Diff}^2_b(\mathcal{M})$. The Laplacian of a cone metric is made up of vector fields

$$X = a_0 \partial_{\mathfrak{r}} + \sum_{j=1}^{n} a_j \frac{1}{\mathfrak{r}} \partial_{x_j}, \quad a_\nu \text{ smooth up to } \partial\mathcal{M}$$

and $\tilde{X} = \mathfrak{r} X$ is a smooth section of ${}^b T\mathcal{M}$. So Laplacians with respect to conic metrics belong to

$$\mathfrak{r}^{-2} \operatorname{Diff}^2_b(\mathcal{M}).$$

Definition

A cone differential operator of order m is an element of $\mathfrak{r}^{-m} \operatorname{Diff}^m_b(\mathcal{M}; E)$.

An element $A \in \mathfrak{r}^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E)$ is in particular a differential operator over the interior of $\mathcal{M}$, so it has a standard principal symbol.

This symbol is singular over $\partial\mathcal{M}$. But one can define another version of the cotangent bundle, the c-cotangent bundle, so that the symbol is smooth up to the boundary.

The c-cotangent bundle

Let $C^\infty_{\mathrm{cn}}(\mathcal{M}; T^*\mathcal{M})$ be the subspace of $C^\infty(\mathcal{M}; T^*\mathcal{M})$ consisting of sections whose pullback to $\partial\mathcal{M}$ vanishes.

There is a vector bundle ${}^cT^*\mathcal{M}$ and vector bundle homomorphism ${}^c\mathrm{ev} : {}^cT^*\mathcal{M} \to T^*\mathcal{M}$ giving an isomorphism

$$ {}^c\mathrm{ev}_* : C^\infty(\mathcal{M}; {}^cT^*\mathcal{M}) \to C^\infty_{\mathrm{cn}}(\mathcal{M}; T^*\mathcal{M}). $$

The composition

$$ {}^c\varpi(A) = \varpi(A) \circ {}^c\mathrm{ev} $$

is smooth. This allows for the definition of ellipticity (cone-ellipticity).

In local coordinates

$$A = \frac{1}{\mathfrak{r}^m} \sum_{k+|\alpha| \le m} a_{k,\alpha}(\mathfrak{r}, x)(\mathfrak{r} D_{\mathfrak{r}})^k D_x^\alpha.$$

The c-symbol is

$${}^c\!\varpi(A) = \sum_{k+|\alpha| = m} a_{k,\alpha}(\mathfrak{r}, x)\rho^k \xi^\alpha.$$

Another pair of symbols also play a role. The "wedge symbol" is an operator on $\partial\mathcal{M} \times \mathbb{R}_+$, locally

$$A_\wedge = \frac{1}{\mathfrak{r}^m} \sum_{k+|\alpha| \le m} a_{k,\alpha}(0, x)(\mathfrak{r} D_{\mathfrak{r}})^k D_x^\alpha.$$

Invariantly defined it is an operator on the inward pointing normal bundle of $\partial\mathcal{M}$. It is c-elliptic if A is.

The other symbol is the "indicial family". Again with A locally written as

$$A = \frac{1}{\mathfrak{r}^m} \sum_{k+|\alpha|\le m} a_{k,\alpha}(\mathfrak{r}, x)(\mathfrak{r} D_{\mathfrak{r}})^k D_x^\alpha.$$

Let $\sigma \in \mathbb{C}$. The value of

$$\mathfrak{r}^m \mathfrak{r}^{-i\sigma} A(\mathfrak{r}^{i\sigma}\phi)$$

restricted to $\partial\mathcal{M}$ depends only of $\phi|_{\partial\mathcal{M}}$. This gives an operator

$$\widehat{A}(\sigma) = \sum_{k+|\alpha|\le m} a_{k,\alpha}(0, x)\sigma^k D_x^\alpha \in \mathrm{Diff}^m(\partial\mathcal{M})$$

(elliptic family depending holomorphically on σ). The boundary spectrum of A is the set

$$\mathrm{spec}_b(A) = \{\sigma \in \mathbb{C} : \widehat{A}(\sigma) \text{ is not invertible}\}.$$

Relevancy of $\operatorname{spec}_b(A)$:

Let A be c-elliptic,

$$\operatorname{spec}_b(A) = \{\sigma_j\}_{j=0}^{\infty}.$$

If $A\phi = 0$, $\phi \in \mathfrak{r}^{-s} L^2_b$, then

$$\phi \sim \sum_{\Im\sigma_j < s} \sum_{k=0}^{N_j} \sum_{\ell=0}^{N_{j,k}} \mathfrak{r}^{i\sigma_j + k} \log^{\ell} \mathfrak{r}\, \phi_{j,k,\ell}$$

with $\phi_{j,k,\ell} \in C^\infty(\partial\mathcal{M}; E)$.[2]
(As for regular singular ODE's, which is why $\widehat{A}$ is called the indicial family.)

[2]This is by now classical. A good reference is
R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, A K Peters, Ltd., Wellesley, MA, 1993.

Domains

The minimal domain of $A \in \mathfrak{r}^{-m} \operatorname{Diff}_b^m(\mathcal{M}; E)$ is the domain of the closure of

$$A : C_c^\infty(\mathring{\mathcal{M}}; E) \subset \mathfrak{r}^{-s} L_b^2(\mathcal{M}; E) \to \mathfrak{r}^{-s} L_b^2(\mathcal{M}; E)$$

This domain, $\mathfrak{D}_{\min}(A)$, is dense in $\mathfrak{r}^{-s} L_b^2$.

One also has the maximal domain,

$$\mathfrak{D}_{\max}(A) = \{u \in \mathfrak{r}^{-s} L_b^2(\mathcal{M}; E) : Au \in \mathfrak{r}^{-s} L_b^2(\mathcal{M}; E)\}$$

The inner product

$$(u, v)_A = (Au, Av) + (u, v) \qquad \text{(the graph norm)}$$

makes $\mathfrak{D}_{\max}$ into a Hilbert space, and $\mathfrak{D}_{\min}$ is a closed subspace (the closure of $C_c^\infty(\mathring{\mathcal{M}}; E)$).

From now on $A \in \mathfrak{r}^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E)$ is c-elliptic and $\mathcal{M}$ is a compact manifold with boundary.

Proposition

[a] *The operators*

$$A : \mathfrak{D}_{\min} \subset \mathfrak{r}^{-s} L^2_b \to \mathfrak{r}^{-s} L^2_b, \qquad A : \mathfrak{D}_{\max} \subset \mathfrak{r}^{-s} L^2_b \to \mathfrak{r}^{-s} L^2_b$$

are Fredholm. Consequently $\mathfrak{D}_{\min}$ *has finite codimension in* $\mathfrak{D}_{\max}$.

[a] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.

Fredholm: both operators have finite-dimensional kernel and closed range of finite codimension. The index is the difference of these numbers. For example

$$\mathrm{Ind}\, A_{\min} = \dim\ker A_{\min} - \dim\operatorname{coker} A_{\min}.$$

Let $\mathcal{E}$ be the orthogonal of $\mathfrak{D}_{\min}$ in $\mathfrak{D}_{\max}$. This is a finite-dimensional space.

Corollary

[a] *Any closed extension of*

$$A : C_c^\infty \subset \mathfrak{r}^{-s} L_b^2 \to \mathfrak{r}^{-s} L_b^2$$

has as domain a subspace $\mathfrak{D} \subset \mathfrak{D}_{\max}$ *of the form*

$$\mathfrak{D} = \mathfrak{D}_{\min} + D$$

for some subspace $D \subset \mathcal{E}$*. The index of* $A_{\mathfrak{D}}$ *is*

$$\operatorname{Ind} A_{\mathfrak{D}} = \operatorname{Ind} A_{\min} + \dim D.$$

[a] Lesch, same place.

Domains are specified by subspaces of $\mathcal{E}$. Let $A^\star$ be the formal adjoint of A ($A^\star \in \mathfrak{r}^{-m}\,\mathrm{Diff}^m_b$ is cone elliptic if and only if A is).

Lemma

$$\mathcal{E} = \ker(A^\star A + I) \cap \mathfrak{D}_{\max}(A).$$

Therefore, if $\phi \in \mathcal{E}$, *then*

$$\phi \sim \sum_{\Im\sigma_j < s} \sum_{k=0}^{N_j} \sum_{\ell=0}^{N_{j,k}} \mathfrak{r}^{i\sigma_j + k} \log^\ell \mathfrak{r}\, \phi_{j,k,\ell}$$

near $\partial\mathcal{M}$ *with* $\phi_{j,k,\ell} \in C^\infty(\partial\mathcal{M}; E)$ *because* $\mathcal{E} \subset \mathfrak{r}^{-s} L^2_b$

Elliptic cone operators on compact manifolds with boundary share properties with

- elliptic operators on compact manifolds without boundary: they are always Fredholm operators;
- elliptic operators on compact manifolds with boundary: boundary conditions specify domains.

Rays of minimal growth

The rest of the exposition concerns existence of rays of minimal growth for elliptic cone operators on compact manifolds. This is joint work with Juan Gil and Thomas Krainer.[3,4,5]

A ray $\Gamma = \{\lambda = re^{i\theta} : r > 0\}$ is a ray of minimal growth for $A_{\mathfrak{D}}$ if there are R and C such that $re^{i\theta} \notin \operatorname{spec}(A_{\mathfrak{D}})$ and the inverse

$$B_{\mathfrak{D}}(\lambda) : \mathfrak{r}^{-s} L^2_b \to \mathfrak{D}$$

of $A_{\mathfrak{D}} - \lambda$ satisfies

$$\|B_{\mathfrak{D}}(\lambda) f\|_A \leq C \|f\|_{\mathfrak{r}^{-s}L^2_b}, \quad f \in \mathfrak{r}^{-s} L^2_b,\ \lambda \in \Gamma,\ |\lambda| > R.$$

[3] J. B. Gil, T. Krainer, G. M., Geometry and spectra of closed extensions of elliptic cone operators, Canad. J. Math. **59** (2007) 742–794.

[4] ________, Resolvents of elliptic cone operators, J. Funct. Anal. **241** (2006) 1–55.

[5] ________, On rays of minimal growth for elliptic cone operators, Oper. Theory Adv. Appl. **172** (2007), 33–50.

For an elliptic differential operator P on a compact manifold without boundary, the condition that Γ is a ray of minimal growth for P is equivalent to[6]

$$\varpi(P) - \lambda \text{ is invertible if } \lambda \in \Gamma.$$

But: assuming ${}^c\varpi(A) - \lambda$ invertible for $\lambda \in \Gamma$ is not sufficient for Γ to be a ray of minimal growth for A: The specific domain and the wedge symbol $A_\wedge$ are involved.

Are there domains such that $\operatorname{res}(A_{\mathfrak{D}}) \neq \emptyset$?

[6] R. Seeley, *Complex powers of an elliptic operator*, Singular Integrals, AMS Proc. Symp. Pure Math. X, 1966, Amer. Math. Soc., Providence, 1967, pp. 288–307.

If

$$A : \mathfrak{D} \subset \mathfrak{r}^{-s} L^2_b \to \mathfrak{r}^{-s} L^2_b$$

is invertible, then $\operatorname{Ind} A_{\mathfrak{D}} = 0$. If $\mathfrak{D} = \mathfrak{D}_{\min} + D$, then

$$\operatorname{Ind} A_{\mathfrak{D}} = \operatorname{Ind} A_{\min} + \dim D$$

so

$$\dim D = -\operatorname{Ind} A_{\min}$$

for such domain. Also, since $\mathfrak{D}_{\max} = \mathfrak{D}_{\min} + \mathcal{E}$,

$$\operatorname{Ind} A_{\mathfrak{D}} = \operatorname{Ind} A_{\min} + \dim D \leq \operatorname{Ind} A_{\min} + \dim \mathcal{E} = \operatorname{Ind} A_{\max}.$$

Lemma

Extensions of index zero exists if and only if $\operatorname{Ind} A_{\min} \leq 0$ *and* $\operatorname{Ind} A_{\max} \geq 0$.

Let $\mathfrak{D} = D + \mathfrak{D}_{\min}$, $\mathfrak{D} \subset \mathcal{E}$. The condition

$$0 = \operatorname{Ind} A_{\mathfrak{D}} = \operatorname{Ind} A_{\min} + \dim D$$

gives $\dim D = -\operatorname{Ind} A_{\min}$.

We let $\mathcal{G}$ be the Grassmannian of $(-\operatorname{Ind} A_{\min})$-dimensional subspaces of $\mathcal{E}$, and let $\mathfrak{G} = \{\mathfrak{D}_{\min} + D : D \in \mathcal{G}\}$.

The elements of $\mathfrak{G}$ are the domains on which A has index 0; its elements are labeled by $\mathcal{G}$, a compact complex manifold.

Assume

$$\operatorname{Ind} A_{\min} < 0 \text{ and } \operatorname{Ind} A_{\max} > 0,$$

so that

$$\mathcal{G} \text{ is neither empty nor one point.}$$

Spectra of extensions

Fix λ. Note: For any $\mathfrak{D}$,

- if $A_{\min} - \lambda$ is not injective, then $A_{\mathfrak{D}} - \lambda$ is not injective;
- if $A_{\max} - \lambda$ is not surjective, then $A_{\mathfrak{D}} - \lambda$ is not surjective.

Definition

The background spectrum of A is

$$\text{bg-spec}(A) = \{\lambda : A_{\min} - \lambda \text{ is not injective and } A_{\max} - \lambda \text{ is not surjective}\}$$

The background resolvent set is $\text{bg-res}(A) = \mathbb{C} \setminus \text{bg-spec}(A)$.

$$\text{bg-res}(A) = \mathbb{C} \setminus \text{bg-spec}(A)$$

The background spectrum is a subset of the spectrum of every extension.

The backround spectrum is a closed set, either all of $\mathbb{C}$, or discrete (using perturbation theory). Assume bg-spec$(A) \neq \mathbb{C}$.
The inclusion map $\mathfrak{D}_{\max} \hookrightarrow L^2_b$ is compact, so

$$\operatorname{Ind}(A_{\mathfrak{D}} - \lambda) = \operatorname{Ind} A_{\mathfrak{D}}$$

for any $\mathfrak{D}$ and λ.
Let $\lambda \in$ bg-res(A). Then $A_{\max} - \lambda$ is surjective. Its kernel has dimension

$$\begin{aligned}\dim\ker(A_{\max} - \lambda) &= \dim\ker(A_{\max} - \lambda) - \dim\operatorname{coker}(A_{\max} - \lambda)\\ &= \operatorname{Ind}(A_{\max} - \lambda) = \operatorname{Ind} A_{\max}\end{aligned}$$

so

$$\mathcal{K}_\lambda = \ker(A_{\max} - \lambda)$$

has constant dimension. The $\mathcal{K}_\lambda$ are viewed as the fibers of a vector bundle

$$\mathcal{K} \to \text{bg-res}(A).$$

This is a holomorphic vector bundle of rank $d' = \operatorname{Ind}(A_{\max})$.

Suppose $\mathfrak{D} = \mathfrak{D}_{\min} + D$, $D \in \mathcal{G}$. Then bg-spec$(A) \subset$ spec$(A_{\mathfrak{D}})$.
Let $\lambda \in$ bg-res(A). Then

$$\begin{aligned}\lambda \in \operatorname{spec}(A_{\mathfrak{D}}) &\iff \exists \phi \in \mathfrak{D},\ \phi \neq 0,\ (A-\lambda)\phi = 0 \\ &\iff \exists \phi \in \mathfrak{D},\ \phi \neq 0,\ \phi \in \mathcal{K}_\lambda \\ &\iff \mathfrak{D} \cap \mathcal{K}_\lambda \neq 0.\end{aligned}$$

Proposition

Let $\mathfrak{D} \in \mathfrak{G}$. *Then*

$$\operatorname{spec}(A_{\mathfrak{D}}) = \text{bg-spec}(A) \cup \{\lambda \in \text{bg-res}(A) : \mathcal{K}_\lambda \cap \mathfrak{D} \neq 0\}.$$

Furthermore, if $\lambda \notin \operatorname{spec}(A_{\mathfrak{D}})$, *then*

$$\mathfrak{D} \oplus \mathcal{K}_\lambda = \mathfrak{D}_{\max}.$$

Note: $\operatorname{Ind} A_{\max} = \operatorname{Ind} A_{\min} + \dim \mathcal{E}$ gives

$$\dim \mathcal{E} = -\operatorname{Ind} A_{\min} + \operatorname{Ind} A_{\max}$$

If $\lambda \in \text{bg-res}(A)$, then $(A_{\max} - \lambda) : \mathfrak{D}_{\max} \to \mathfrak{r}^{-s} L^2_b$ is continuous and surjective. The operator

$$(A - \lambda) : \mathcal{K}_\lambda^\perp \to \mathfrak{r}^{-s} L^2_b$$

is invertible. Let

$$B_{\max}(\lambda) : \mathfrak{r}^{-s} L^2_b \to \mathfrak{D}_{\max}$$

be the inverse.

If $\lambda \notin \text{spec}(A_{\mathfrak{D}})$ then $\mathfrak{D}_{\max} = \mathcal{K}_\lambda \oplus \mathfrak{D}$. Let

$$\pi_{\mathcal{K}_\lambda,\mathfrak{D}} = \text{ projection on } \mathcal{K}_\lambda \text{ according to } \mathfrak{D}_{\max} = \mathcal{K}_\lambda \oplus \mathfrak{D}.$$

The resolvent of $A_{\mathfrak{D}}$ is

$$B_{\mathfrak{D}}(\lambda) = B_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda,\mathfrak{D}} B_{\max}(\lambda).$$

Let R be a smooth real vector field on $\mathcal{M}$ with $R\mathfrak{r} = 1$ near $\partial\mathcal{M}$. Let χ_t be its flow, and let

$$\kappa_\varrho u = f_\varrho (u \circ \chi_{\log \varrho}), \varrho > 0.$$

The function f_ϱ is chosen so that

$$\kappa_\varrho : \mathfrak{r}^{-s} L^2_b \to \mathfrak{r}^{-s} L^2_b \quad \text{is an isometry.}$$

In coordinates $(x, \mathfrak{r})$ near $\partial\mathcal{M}$, if $R = \partial_\mathfrak{r}$, then

$$(\kappa_\varrho u)(x, \mathfrak{r}) = \mathfrak{r}^{s/2} u(x, \varrho\mathfrak{r}).$$

The operator $A_\wedge$ is obtained from A using κ:

$$A_\wedge = \lim_{\varrho \to 0} \varrho^m \kappa_\varrho A \kappa_\varrho^{-1}$$

The operator $A_\wedge$ is defined on $\partial\mathcal{M} \times \mathbb{R}_+$. It has its own minimal and maximal domains, background spectrum and resolvent set.

$$\text{bg-spec}(A_\wedge) \text{ is a union of closed sectors in } \mathbb{C}.$$

$A_\wedge$ is not Fredholm with any domain, but if $\lambda \in \text{bg-res}(A)$, then $A - \lambda$ is Fredholm.

On each connected component Λ of $\text{bg-res}(A_\wedge)$ there is a complex vector bundle $\mathcal{K}_\wedge \to \Lambda$,

$$\mathcal{K}_{\wedge,\lambda} = \ker(A_{\wedge,\max} - \lambda)$$

An example of this is the Laplacian on $\mathbb{R}^2 \backslash 0$. Its background spectrum is $\overline{\mathbb{R}}_+$, and $\Delta - \lambda$ is Fredholm if $\lambda \notin \overline{\mathbb{R}}_+$

There is a natural bijection

$$\theta : \mathfrak{G} \to \mathfrak{G}_\wedge$$

from domains for A to domains for $A_\wedge$.

Two theorems

Theorem

Let $\mathfrak{D}$ be a domain for A of index 0. Let Γ be a ray in $\mathbb{C}$. Suppose that

$$^c\!\varpi(A)(\boldsymbol{\xi}) - \lambda \text{ is invertible for every } \lambda \in \Gamma \text{ and } \boldsymbol{\xi} \in {}^cT^*\mathcal{M}\backslash 0.$$

Suppose further that

$$\Gamma \text{ is a ray of minimal growth for } A_\wedge \text{ with domain } \theta(\mathfrak{D})$$

Then Γ is a ray of minimal growth for $A_{\mathfrak{D}}$.

The vector field $\mathfrak{r}\partial_\mathfrak{r}$ on $\partial\mathcal{M} \times \mathbb{R}_+$ gives a unitary map

$$\kappa_{\wedge,\varrho} : \mathfrak{r}^{-s} L^2_b(\partial\mathcal{M} \times \mathbb{R}_+) \to \mathfrak{r}^{-s} L^2_b(\partial\mathcal{M} \times \mathbb{R}_+).$$

Let $\pi_{\max} : \mathfrak{D}_{\wedge,\max} \to \mathfrak{D}_{\wedge,\max}$ be the orthogonal projection on $\mathcal{E}_\wedge =$ orthogonal of $\mathfrak{D}_{\wedge,\min}$ in $\mathfrak{D}_{\wedge,\max}$.
κ_ϱ acts on subspaces D of $\mathcal{E}_\wedge$ via $\pi_{\wedge,\max}\kappa_{\wedge,\varrho}D$. This is an $\mathbb{R}_+$ action.
Given a subspace $D \subset \mathcal{E}_\wedge$, let $\Omega^-(D)$ be the set of spaces D'such that there is a sequence $\{\varrho_\nu\}$, $\varrho_\nu \to 0$, such that $\pi_{\wedge,\max}\kappa_{\wedge,\varrho_\nu} \to D'$ (the α-limit set of D).
Let $\lambda_0 \in \text{bg-res}(A_\wedge)$. Let $\mathcal{G}_\wedge$ be the Grassmannian of subspaces of $\mathcal{E}_\wedge$ such that $A_\wedge - \lambda_0$ with domain $D + \mathfrak{D}_{\wedge,\min}$ has index 0.
Let $\mathfrak{V} = \{D \in \mathcal{G}_\wedge : D \cap \mathcal{K}_{\wedge,\lambda_0} \neq 0\}$.

Theorem

The ray through λ_0 is a ray of minimal growth for $A_\wedge$ with domain $\mathfrak{D}_{\wedge,\min} + D$ if and only if $\mathfrak{V} \cap \Omega^-(D) = \emptyset$.