Singular foliations by tori

. . . with some additional structure

Gerardo Mendoza

Microlocal and Global Analysis, Interactions with Geometry

Potsdam, March 2019

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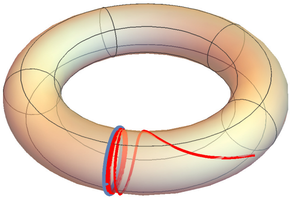

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The next few slides give some motivation (why look at this?), then state a classification theorem for pairs $(\mathcal{N}, \mathcal{T})$ similar to the classification of line bundles by their Chern class, then do some analysis.

Where did this come from

Suppose $\mathcal{M}$ is a compact manifold with boundary $\mathcal{N}$. Suppose that its b-tangent bundle comes with an almost complex structure J:

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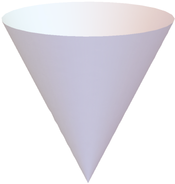

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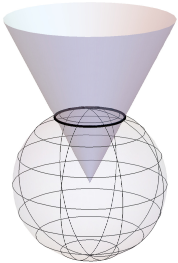

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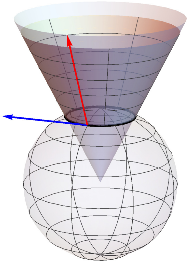

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In the context of complex *b*-structures, the boundary also has a canonical vector subbundle $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ inherited form the subbundle ${}^bT^{0,1}\mathcal{M}$ of which $\mathcal{T}$ is a section.

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So looking at $\mathcal{N}$ with $\mathcal{T}$ and possibly additional structure generalizes some parts of complex geometry in a non-standard direction.

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$(\mathcal{N}, \mathcal{T})$, $(\mathcal{N}', \mathcal{T}') \in \mathscr{F}$ are:

– **Globally equivalent** if there is an equivariant diffeomorphism $h : \mathcal{N}' \to \mathcal{N}$
– **Locally equivalent** if there are open covers $\{U_a\}_{a \in A}$ of $\mathcal{N}$ and $\{U'_a\}_{a \in A}$ of $\mathcal{N}'$ by $\mathcal{T}$, resp. $\mathcal{T}'$-invariant open sets and equivariant diffeomorphisms $h_a : U'_a \to U_a$ for each $a \in A$ such that

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If $(\mathcal{N}, \mathcal{T})$, $(\mathcal{N}', \mathcal{T}') \in \mathscr{F}$ are locally equivalent, then $\mathcal{B}_{\mathcal{N}}$ and $\mathcal{B}_{\mathcal{N}'}$ are homeomorphic.

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(*) $h_{ab} = h_a h_b^{-1}(p) \in \overline{\mathcal{O}}_p$ for every $a, b \in A$ and $p \in U_a \cap U_b$.

Let $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$. There is an exact sequence

$$0 \to \mathscr{Z} \xrightarrow{\iota} \mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathscr{I}^\infty(\mathcal{N}) \to 0$$

of sheaves in which every $(\mathcal{N}', \mathcal{T}')$ which is locally equivalent to $(\mathcal{N}, \mathcal{T})$ defines an element in the first Čech cohomology group $\check{H}^1(\mathcal{B}; \mathscr{I}^\infty(\mathcal{N}))$ via (*).

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– $\mathcal{I}^\infty(\mathcal{N}_V)$ be the set of smooth $\mathcal{T}$-equivariant diffeomorphisms

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– $C^\infty(V, \mathfrak{g})$ be the space of smooth functions $\mathcal{N}_V \to \mathfrak{g}$ which are constant on the orbits of $\mathcal{T}$ and $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g})$ is the corresponding sheaf.

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$$0 \to \mathscr{Z} \xrightarrow{\iota} \mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathscr{I}^\infty(\mathcal{N}) \to 0$$

of sheaves in which every $(\mathcal{N}', \mathcal{T}')$ which is locally equivalent to $(\mathcal{N}, \mathcal{T})$ defines an element in the first Čech cohomology group $\check{H}^1(\mathcal{B}; \mathscr{I}^\infty(\mathcal{N}))$ via (*). For $V \subset \mathcal{B}$ open let $\mathcal{N}_V$ be the part of $\mathcal{N}$ over $\mathcal{B}$.

– $\mathcal{I}^\infty(\mathcal{N}_V)$ be the set of smooth $\mathcal{T}$-equivariant diffeomorphisms

$$h : \mathcal{N}_V \to \mathcal{N}_V \text{ such that } h(p) \in \overline{\mathcal{O}}_p \text{ for all } p \in \mathcal{N}_V$$

$\mathscr{I}^\infty(\mathcal{N}) \to 0$ is the corresponding sheaf.

– $C^\infty(V, \mathfrak{g})$ be the space of smooth functions $\mathcal{N}_V \to \mathfrak{g}$ which are constant on the orbits of $\mathcal{T}$ and $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g})$ is the corresponding sheaf.

$\mathscr{Z} \to \mathcal{B}$ be the sheaf of locally constant $\mathfrak{z}$-valued functions on $\mathcal{B}$, with

$$\mathfrak{z} = \ker(\exp : \mathfrak{g} \to G).$$

$\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g})$ is a fine sheaf, so the long exact sequence in cohomology gives an isomorphism $\check{H}^1(\mathcal{B}, \mathscr{I}^\infty(\mathcal{N})) \to \check{H}^2(\mathcal{B}, \mathscr{Z}) \approx H^2(\mathcal{B}, \mathbb{Z}^d)$ ($d = \dim G$).

Classification, cont.

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There is a bijection between the elements of $\check{H}^2(\mathcal{B}, \mathscr{Z})$ and the global equivalence classes of elements of $\mathscr{F}$ which are locally equivalent to $\mathcal{N}$.

Gysin sequence

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in which $H^q_{\mathrm{dR}}(\mathcal{N})$ are the de Rham cohomology groups of $\mathcal{N}$, the groups $H^q_{\mathrm{dR}}(\mathcal{B})$ are the cohomology groups of the complex

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What is $\mathbf{e}$?

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Let $\langle \theta, v\rangle = g(\mathcal{T}, v)$. Then $\mathcal{L}_{\mathcal{T}} d\theta = 0$, so $d\theta \in C^\infty(\mathcal{B}; \bigwedge^2 \mathcal{B})$: $\mathbf{e} = [d\theta]$
If $\phi \in C^\infty(\mathcal{B}; \bigwedge^{q-2} \mathcal{B})$ and $d\phi = 0$ then $d\theta \wedge \phi \in C^\infty(\mathcal{B}; \bigwedge^q \mathcal{B})$ and $d(d\theta \wedge \phi) = 0$.

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ρ_* is such that $\rho_*\phi$ and $d\phi = 0$ implies $\phi = d\theta \wedge \psi$

Embeddings

Let $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$. Then there is a positive integer N, an embedding

$$F : \mathcal{N} \to \mathbb{C}^N$$

with image contained in the sphere S^{2N-1}, and positive numbers τ_j such that

$$F_* \mathcal{T} = i \sum_{j=1}^{N} \tau_j \left(z^j \frac{\partial}{\partial z^j} - \overline{z}^j \frac{\partial}{\partial \overline{z}^j} \right).$$

No component of F is flat at any point of $\mathcal{N}$.

Embeddings

Let $(\mathcal{N},\mathcal{T}) \in \mathscr{F}$. Then there is a positive integer N, an embedding

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The closure of $\mathfrak{a}_t$ in the group of isometries of $\mathcal{N}$ is isomorphic to the closure of

$$\{(e^{i\tau_1 t}, e^{i\tau_2 t}, \dots, e^{i\tau_N t}) : t \in \mathbb{R}\} \subset S^1 \times S^1 \cdots \times S^1.$$

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$$\mathcal{E}_{\tau,\lambda} = \{\phi \in C^\infty(\mathcal{N}) : -i\mathcal{T}\phi = \tau\phi,\ \Delta\phi = \lambda\phi\}$$

The set $\mathrm{sp}(-i\mathcal{T},\Delta) = \{(\tau,\lambda) : \mathcal{E}_{\tau,\lambda} \neq 0\}$ is a discrete subset of $\mathbb{R}^2$.

The map $F = (f^1,\dots,f^N) : \mathcal{N} \to \mathbb{C}^N$ satisfies $F_*\mathcal{T} = \mathcal{T}'$ iff $\mathcal{T}f^j = i\tau_j f^j$ which suggests using (non-zero) functions in the $\mathcal{E}_{\tau,\lambda}$ as f^j's.

Let $\phi_{\tau,\lambda,j}$, $j = 1,\dots,N_{\tau,\lambda}$, be an orthonormal basis of $\mathcal{E}_{\tau,\lambda}$

Properties:

1. $\mathbb{C}T^*_{p_0}\mathcal{N} = \mathrm{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau,\lambda) \in \mathrm{sp}(-i\mathcal{T},\Delta),\ j = 1,\dots,N_{\tau,\lambda}\}$;
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$\Delta\overline{\phi} = \overline{\Delta\phi}$ gives $(\tau,\lambda) \in \mathrm{sp}(-i\mathcal{T},\Delta) \implies (-\tau,\lambda) \in \mathrm{sp}(-i\mathcal{T},\Delta)$: can take all τ of the same sign.

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