Domains of closed extensions of ODEs

Part II

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We say that A is symmetric if $A^\star = A$, that is,

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$\dim \mathcal{E}$ even is necessary in order to have a lagrangian space with respect to $[\cdot, \cdot]_A$

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Let $\delta = \min\{\dim \mathcal{K}_\lambda\}$. This is a finite number.

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$$\mathcal{R}_\lambda = \ker(A^\star_{\max} - \overline{\lambda}\mathrm{I})$$

and $A^\star = A$, so $A^\star_{\max} - \overline{\lambda} = A_{\max} - \overline{\lambda}$. We conclude $\mathcal{K}_\lambda = \mathcal{R}_{\overline{\lambda}}$ and

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In particular

The spaces $\ker(A_{\max} - i)$, $\ker(A_{\max} + i)$ have the same dimension.

By a theorem of von Neumann,[1] $A_{\min}$ has a selfadjoint extension. Lesch[2] has an analysis of selfadjoint extensions from the perspective of the von Neumann theory.

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Proof: 1. $D_T^\perp = \{(-T^*w + w : w \in D_0^\perp\}$

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So $-T^* A = AT$. But $(AT)^* = T^* A^* = -T^* A$. $\square$

The structure of $\mathfrak{SA}$

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The structure of $\mathfrak{SA}$: $\mathfrak{SA}$ is a real-analytic submanifold of $\mathrm{Gr}_{d'}(\mathcal{E})$.

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