Domains of closed extensions of ODEs

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- to then discus the set of domains of closed extensions when the operator admits more than one such extension and of selfadjoint extensions for symmetric semibounded operators.

- and finally to discuss the behavior of the eigenvalues as the domain varies over the space of selfadjoint extensions.

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$[v_{j,-1}, v_{j,1}]$

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$$A_j = \frac{1}{(1-x^2)^m} \sum_{k=0}^{m} a_{j,k}(x) \big((1-x^2)D_x\big)^k$$

where D_x means $-id/dx$.

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A_j acts on vectors $u(x) = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix}$

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On each interval I_j (viewed as $[-1,1]$) we do the following. We start with $C_c^\infty((-1,1),\mathbb{C}^n)$ with the inner product

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$\mathcal{M}$ is the union of the I_j.

Set-up: the spaces, continuation

Next we let

"the graph inner product."

$$(u, v)_{A_j} = (A_j u, A_j v) + (u, v) \quad u, v \in C_c^\infty((-1,1), \mathbb{C}^n),$$

another inner product on $C_c^\infty((-1,1), \mathbb{C}^n)$,

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- $\mathcal{D}_{\min}(A) = \bigoplus_{j=1}^N \mathcal{D}_{\min}(A_j)$

Set-up: the spaces, end

And now

let $\mathcal{D}_{\max}(A_j)$ be the subspace of all $u \in L^2((-1,1),\mathbb{C}^n)$ such that there is $f \in L^2((-1,1),\mathbb{C}^n)$ such that

$$(u, A_j^\star \phi) = (f, \phi) \quad \textit{for all } \phi \in C_c^\infty((-1,1),\mathbb{C}^n).$$

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The f with this property is unique. If also f' works, then $(f - f', \phi) = 0$ for all $\phi \in C_c^\infty$. Since C_c^∞ is dense in L^2, $f - f'$ must vanish. Define $A_j u = f$. Since $\mathcal{D}_{\min} \subset \mathcal{D}_{\max}$, this defines an extension of A_j.

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In turn this space, $\mathcal{E} = \mathcal{D}_{\min}^{\perp} \subset \mathcal{D}_{\max}$, is equal to

$$\ker(A^{\star}A + \mathrm{I}) \cap \mathcal{D}_{\max}(A).$$

The important consequence of this is that every closed extension of A has as domain a subspace $\mathcal{D} \subset \mathcal{D}_{\max}$ of the form

$$\mathcal{D} = D + \mathcal{D}_{\min}, \quad D \subset \mathcal{E}.$$

For this reason it is of interest to study the structure of the set of subspaces of $\mathcal{E}$ of a given dimension.

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The extra structure is that

> $\mathrm{Gr}_k(\mathcal{E})$ *is a* complex manifold *(of complex dimension $d_k = k \times (d - k)$)*

This means that $\mathrm{Gr}_k(\mathcal{E})$ is locally like $\mathbb{C}^{d_k}$.

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Let $D_0 \in \mathrm{Gr}_k(\mathcal{E})$ and let $T : D_0 \to D_0^\perp$ be a linear map ($D_0^\perp \in \mathrm{Gr}_{d-k}(\mathcal{E})$). Define

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Define, for $\varepsilon > 0$,

$$U(D_0, \varepsilon) = \{D_T : T \in \mathcal{L}(D_0, D_0^\perp), \|T\| \le \varepsilon\}$$

The family $\{U(D_0, \varepsilon) : D_0 \in \mathrm{Gr}_k(\mathcal{E}), \varepsilon > 0\}$ is a base for a topology of $Gr_k(\mathcal{E})$.

A subset $U \subset \mathrm{Gr}_k(\mathcal{E})$ is open iff for all $D_0 \in U$ there is $\varepsilon > 0$ such that $U_{D,\varepsilon} \subset U$.

Fix an orthonormal basis $\mathbf{u} = (u_1, \ldots, u_k)$ of D_0 and another one, $\mathbf{v} = (v_1, \ldots, v_{d-k})$, of $D_0^\perp$. Then we have

$$Tu_j = \sum_{i=1}^{d-k} a^i_j v_i, \quad j = 1, \ldots, k, \text{ some } a^i_j \in \mathbb{C}.$$

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Let $U(D_0) = \{D_T : T \in \mathcal{L}(D_0, D_0^\perp)\}$. Fixing bases $\mathbf{u}$, $\mathbf{v}$ as above we get a **bijective** map $\phi_{D_0,\mathbf{u},\mathbf{v}} : U(D_0) \to \mathfrak{gl}(d, \mathbb{C})$. Pick another choice of $D_0', \mathbf{u}', \mathbf{v}'$. We get that

$$\phi_{D_0,\mathbf{u},\mathbf{v}} \circ \phi_{D_0',\mathbf{u}',\mathbf{v}'} : \phi_{D_0',\mathbf{u}',\mathbf{v}'}\big(U(D_0') \cap U(D_0)\big) \to \phi_{D_0,\mathbf{u},\mathbf{v}}\big(U(D_0) \cap U(D_0)\big)$$

is a holomorphic map.

This is what makes $\mathrm{Gr}_k(\mathcal{E})$ a complex manifold.

Why do we care?

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Because this gives useful ways of expressing ideas, eventually leading to a better understanding of domains and of spectra.

Adjoints

Hilbert spaces (annotating H and H'); the graph of T is closed (annotating "closed"); D is dense in H

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By definition,

$$T^* v = f.$$

This defines

$$T^* : D^* \subset H' \to H.$$

T^* is linear, closed, densely defined.

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$$\mathcal{D}_{\max}(A) = \{u \in L^2 : \exists f \in L^2 \text{ s.t.} (u, A^\star\phi) = (f, \phi)\ \forall \phi \in C_c^\infty\}$$

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This definition gives:

If $u \in \mathcal{D}_{\max}$, then $(u, A^\star v) = (Au, v)$ for all $u \in \mathcal{D}_{\max}(A)$ and $v \in \mathcal{D}_{\min}(A^\star)$.

As a consequence $A_{\max}$ is the adjoint of $A^\star_{\min}$.

In general, let $\mathcal{D} = D + \mathcal{D}_{\min}(A)$ be some domain for A. Then

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That is:

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So

$\mathcal{D}_{\min}(A) \ni \phi \mapsto (f, A\phi) \in \mathbb{C}$ is continuous and therefore $f \in \mathcal{D}_{\max}(A^\star)$, and in addition, $A^\star f = -u$.

This gives $A^\star A u = -u$, that is, $(A^\star A + I)u = 0$.

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In other words,

$$u \in \mathcal{E} \implies A^\star u \in \mathcal{E}^\star.$$

A defines a bijective map

$$A|_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\star.$$

Its inverse is

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The only thing not verified already is the last statement. This we do as follows. Let $u, v \in \mathcal{E}$.

$$(Au, Av)_{A^\star} = (A^\star Au, A^\star Av) + (Au, Av) = (Au, Av) + (u, v) = (u, v)_A.$$

Proposition: *Let $\mathcal{D} = D + \mathcal{D}_{\min}$. The domain of the adjoint of $A_{\mathcal{D}}$ is $A^\star$ with domain $D^* + \mathcal{D}_{\min}(A^\star)$, where*

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replace $Au = w$ and $u = -A^\star w$. Get:

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