Elliptic operators on manifolds with conical singularities, II

Gerardo A. Mendoza

Temple University

Paderborn, May 26, 2011

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a b-differential operator of order m if

$$x^{-\nu} P x^{\nu} \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

is a b-differential operator of order m if

$$x^{-\nu} P x^{\nu} \in \operatorname{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

is a b-differential operator of order m if

$\mathrm{Diff}^m(\mathcal{M}; E, F)$ *is the space of linear differential operators of order m with smooth coefficients.*

$$x^{-\nu} P x^\nu \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

$\mathrm{Diff}^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order m with smooth coefficients.

is a b-differential operator of order m if

$$x^{-\nu} P x^{\nu} \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

$\mathrm{Diff}^m_b(\mathcal{M}; E, F)$ is the space of linear b-differential operators on $\mathcal{M}$ of order m with smooth coefficients.

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

is a b-differential operator of order m if

$\mathrm{Diff}^m(\mathcal{M}; E, F)$ *is the space of linear differential operators of order m with smooth coefficients.*

$$x^{-\nu} P x^{\nu} \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

$\mathrm{Diff}_b^m(\mathcal{M}; E, F)$ *is the space of linear b-differential operators on $\mathcal{M}$ of order m with smooth coefficients.*

Near $\partial\mathcal{M}$, P has the form

$$\sum_{\alpha+k\leq m} a_{k\alpha}(xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.$$

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

is a b-differential operator of order m if

$\mathrm{Diff}^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order m with smooth coefficients.

$$x^{-\nu} P x^\nu \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

$\mathrm{Diff}_b^m(\mathcal{M}; E, F)$ *is the space of linear b-differential operators on $\mathcal{M}$ of order m with smooth coefficients.*

Near $\partial\mathcal{M}$, P has the form

$$\sum_{\alpha+k\le m} a_{k\alpha}(xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.$$

The b-symbol of P is

$${}^b\boldsymbol{\sigma}(P) = \sum_{\alpha+k\le m} a_{k\alpha}\xi^k\eta^\alpha.$$

${}^b\boldsymbol{\sigma}(P)$ is a section of ${}^b\pi^*\mathrm{Hom}(E, F)$
${}^b\pi : {}^bT^*\mathcal{M} \to \mathcal{M}$

b-Differential operators, b-symbol, b-ellipticity

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

is a b-differential operator of order m if

$\mathrm{Diff}^m(\mathcal{M}; E, F)$ *is the space of linear differential operators of order m with smooth coefficients.*

$$x^{-\nu} P x^\nu \in \mathrm{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

$\mathrm{Diff}^m_b(\mathcal{M}; E, F)$ *is the space of linear b-differential operators on $\mathcal{M}$ of order m with smooth coefficients.*

Near $\partial\mathcal{M}$, P has the form

$$\sum_{\alpha+k\le m} a_{k\alpha}(xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.$$

The b-symbol of P is

$${}^b\boldsymbol{\sigma}(P) = \sum_{\alpha+k\le m} a_{k\alpha}\xi^k\eta^\alpha.$$

${}^b\boldsymbol{\sigma}(P)$ is a section of ${}^b\pi^*\mathrm{Hom}(E, F)$
${}^b\pi : {}^bT^*\mathcal{M} \to \mathcal{M}$

P is b-elliptic if ${}^b\boldsymbol{\sigma}(P)$ is invertible for $(\xi, \eta) \ne 0$.

Mellin transform

If $\mathcal{M}$ is a manifold with boundary, then

$$\dot{C}^\infty(M) = \{u \in C^\infty(\mathcal{M}) : u \text{ vanishes to infinite order on } \partial\mathcal{M}\}$$

Let $u \in \dot{C}^\infty[0,\infty)$ be compactly supported. The Mellin transform of u is

$$\mathscr{M}(u)(\sigma) = \int_0^\infty x^{-i\sigma} u(x)\, \frac{dx}{x}$$

The inverse Mellin transform is

$$\mathscr{M}^{-1}(v) = \frac{1}{2\pi} \int_{\mathrm{Re}\,\sigma=0} x^{i\sigma} v(\sigma)\, d\sigma.$$

The Mellin transform is the Fourier transform with e^t replaced by x.

If $P \in \mathrm{Diff}^m_b(\mathcal{M}; E, F)$, then P defines an operator P_b on $\mathcal{N} = \partial\mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of u, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$.

If $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$, then P defines an operator P_b on $\mathcal{N} = \partial\mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of u, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$. The indicial operator of $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ is

$$\widehat{P}(\sigma) = (x^{-i\sigma} P x^{i\sigma})_b$$

If $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, then P defines an operator P_b on $\mathcal{N} = \partial\mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of u, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$. The indicial operator of $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$ is

$$\widehat{P}(\sigma) = (x^{-i\sigma} P x^{i\sigma})_b$$

$\widehat{P}(\sigma)$ is a polynomial in $\sigma \in \mathbb{C}$ with values in $\text{Diff}^m(\mathcal{N}; E, F)$. Locally,

$$\widehat{P}(\sigma) = \sum_{k+|\alpha| \le m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$

If $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$, then P defines an operator P_b on $\mathcal{N} = \partial\mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of u, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$. The indicial operator of $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ is

$$\widehat{P}(\sigma) = (x^{-i\sigma} P x^{i\sigma})_b$$

$\widehat{P}(\sigma)$ is a polynomial in $\sigma \in \mathbb{C}$ with values in $\mathrm{Diff}^m(\mathcal{N}; E, F)$. Locally,

$$\widehat{P}(\sigma) = \sum_{k+|\alpha| \le m} a_{k\alpha}(0, y) \sigma^k D_y^\alpha.$$

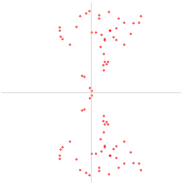
If P is elliptic, then $\widehat{P}(\sigma)$ is elliptic. In this case, $\widehat{P}(\sigma)$ is a holomorphic Fredholm family of index 0. The set

$$\mathrm{spec}_b(P) = \{\sigma \in \mathbb{C} : \widehat{P}(\sigma) \text{ is not injective}\}$$

is discrete,

$$\mathrm{spec}_b(P) \cap \{\sigma : |\sigma| < a\}$$

is finite for every $a > 0$.



Sobolev spaces

1. If s is a nonnegative integer, then $H^s_b(\mathcal{M})$ consists of all $u \in L^2_b(\mathcal{M})$ such that

$$X_1 \dots X_k u \in L^2_b(\mathcal{M}) \text{ for all } X_1, \dots, X_k \in C^\infty(\mathcal{M}; {}^bT^*\mathcal{M}), k \le s$$

2. The space $H^{-s}_b(\mathcal{M})$ is the dual of $H^s_b(\mathcal{M})$
3. If s is not an integer, then $H^s_b(\mathcal{M})$ is defined by interpolation.
4. If s and ν are real numbers, then $x^\nu H^s_b(\mathcal{M}) = \{x^\nu u : u \in H^s_b(\mathcal{M})\}$.

If $P \in \mathrm{Diff}^m_b(\mathcal{M}; E, F)$, then $P : x^\nu H^s_b(\mathcal{M}; E) \to x^\nu H^{s-m}_b(\mathcal{M}; F)$ is continuous.

Elliptic regularity

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ *be b-elliptic. If* $u \in x^\nu H_b^s(\mathcal{M}; E)$ *and* $Pu \in x^\nu H_b^s(\mathcal{M}; F)$*, then* $u \in x^\nu H_b^{s+m}(\mathcal{M}; E)$*.*

The proof is by construction of an operator Q such that

$$QP = \mathrm{I} - R$$

with $Q : x^\nu H_b^t \to x^\nu H_b^{t+m}$ and $R : x^\nu H_b^t \to x^\nu H_b^\infty$ for any t.

Elliptic regularity

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ be b-elliptic. If $u \in x^\nu H_b^s(\mathcal{M}; E)$ and $Pu \in x^\nu H_b^s(\mathcal{M}; F)$, then $u \in x^\nu H_b^{s+m}(\mathcal{M}; E)$.

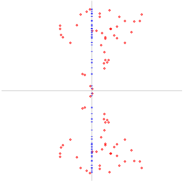
The proof is by construction of an operator Q such that

$$QP = \mathrm{I} - R$$

with $Q : x^\nu H_b^t \to x^\nu H_b^{t+m}$ and $R : x^\nu H_b^t \to x^\nu H_b^\infty$ for any t.

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ be b-elliptic. The map $P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F)$ is Fredholm if and only if $-\nu \notin \{\mathrm{Im}\,\sigma : \sigma \in \mathrm{spec}_b(P)\}$

$$P : x^\nu H_b^{s+m}(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$
$$\iff -\nu \notin \{\mathrm{Im}\,\sigma : \sigma \in \mathrm{spec}_b(P)\}$$



Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $-\nu \notin \mathrm{Im}\ \mathrm{spec}_b(P)$. To prove

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $-\nu \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact.

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $-\nu \notin \operatorname{Im} \mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact.

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \operatorname{Im} \operatorname{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$
defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.

$$\breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

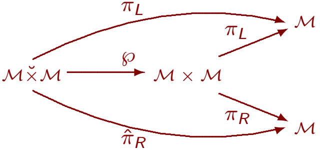
Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M}\breve{\times}\mathcal{M}$ supported near $\breve{\Delta}$ s.t.

$$\breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.



$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x')$

$= \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

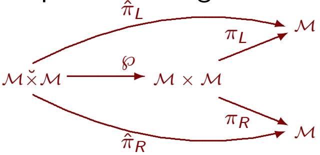
$r = \frac{x + x'}{2}$, $s = \frac{x - x'}{x + x'}$

Suppose $P \in \mathrm{Diff}^m_b$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H^s_b(\mathcal{M}; E) \to H^s_b(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.



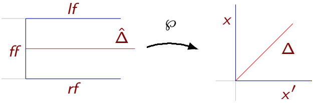
$$\breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x')$

$$= \Big(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\Big)$$

$$r = \frac{x + x'}{2},\ s = \frac{x - x'}{x + x'}$$

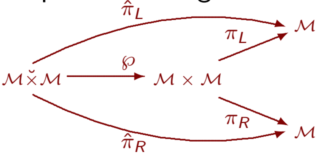


Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.



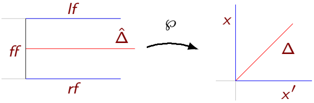
$$\breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

$$\hat{\pi}_L^* P = \sum_{k+|\alpha| \le m} a_{k,\alpha}((1+s)r, y)\left(\tfrac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$$

$$r = \frac{x + x'}{2}, \; s = \frac{x - x'}{x + x'}$$



Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

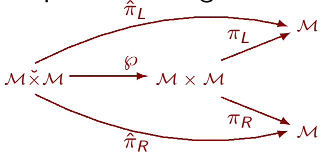
$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.

$$\breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

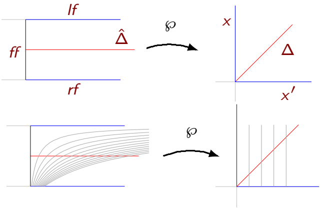
with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$$\hat{\pi}_L^* P = \sum_{k+|\alpha| \le m} a_{k,\alpha}((1+s)r, y)\left(\tfrac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$$



$$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$$

$$r = \frac{x + x'}{2}, \; s = \frac{x - x'}{x + x'}$$

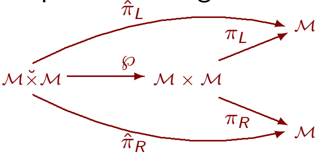


Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*) \quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\dfrac{r(1+s)}{2}, \dfrac{r(1-s)}{2}\right)$

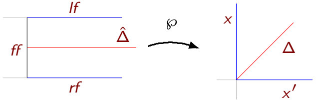
$r = \dfrac{x + x'}{2}$, $s = \dfrac{x - x'}{x + x'}$

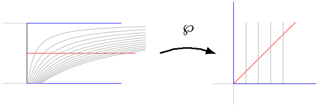
Try

$\hat{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \ldots, \sigma, \eta)\, d\sigma\, d\eta$$

Without much work, get $(*)$ with smooth $\breve{K}_R$.



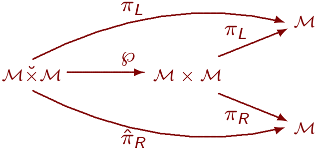


Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \operatorname{Im} \operatorname{spec}_b(P)$. To prove

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

Suppose $\nu = 0$.

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*) \quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

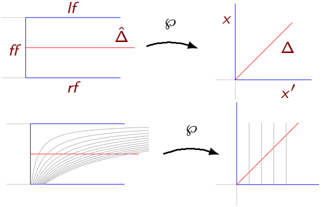
$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

Try

$\hat{\pi}_L^* P = \sum_{k+|\alpha| \le m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \ldots, \sigma, \eta)\, d\sigma\, d\eta$$

$r = \frac{x + x'}{2}, \; s = \frac{x - x'}{x + x'}$



Without much work, get $(*)$ with smooth $\breve{K}_R$.

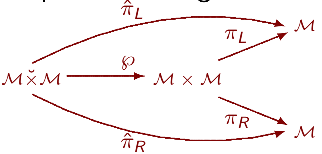
Next, find $\breve{K}_{Q_1}$ such that $\breve{\pi}_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

Suppose $P \in \mathrm{Diff}^m_b$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H^s_b(\mathcal{M}; E) \to H^s_b(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M}\breve{\times}\mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*)\quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

Try

$\hat{\pi}_L^* P = \sum_{k+|\alpha|\le m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

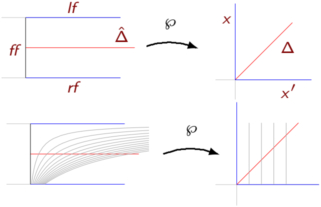
$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \ldots, \sigma, \eta)\, d\sigma\, d\eta$$

$r = \frac{x + x'}{2}$, $s = \frac{x - x'}{x + x'}$

Without much work, get $(*)$ with smooth $\breve{K}_R$.

Next, find $\breve{K}_{Q_1}$ such that $\breve{\pi}_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

$$\breve{\pi}_L^* P|_{ff} = \sum_{k+|\alpha|\le m} a_{k,\alpha}(0, y)\left(\tfrac{1}{2}(1 - s^2)D_s\right)^k D_y^\alpha$$

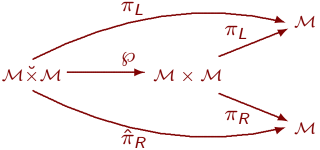


Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M}\breve{\times}\mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*) \quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

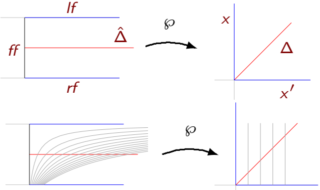
$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

Try

$\hat{\pi}_L^* P = \sum_{k+|\alpha|\le m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

$r = \dfrac{x + x'}{2}$, $s = \dfrac{x - x'}{x + x'}$

$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \dots, \sigma, \eta)\, d\sigma\, d\eta$$



Without much work, get $(*)$ with smooth $\breve{K}_R$.

Next, find $\breve{K}_{Q_1}$ such that $\breve{\pi}_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

$$\breve{\pi}_L^* P|_{ff} = \sum_{k+|\alpha|\le m} a_{k,\alpha}(0, y)\left(\tfrac{1}{2}(1 - s^2)D_s\right)^k D_y^\alpha$$

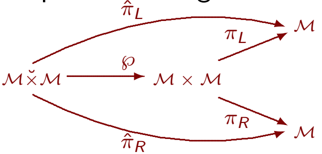
$s = \tanh(t/2) \implies D_t = \frac{1}{2}(1 - s^2)D_s$

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*)\quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

$r = \frac{x + x'}{2}$, $s = \frac{x - x'}{x + x'}$

Try

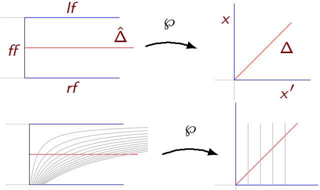
$\hat{\pi}_L^* P = \sum_{k+|\alpha|\le m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \dots, \sigma, \eta)\, d\sigma\, d\eta$$

Without much work, get $(*)$ with smooth $\breve{K}_R$.

Next, find $\breve{K}_{Q_1}$ such that $\breve{\pi}_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

$$\breve{\pi}_L^* P|_{ff} = \sum_{k+|\alpha|\le m} a_{k,\alpha}(0, y)\left(D_t \right)^k D_y^\alpha$$



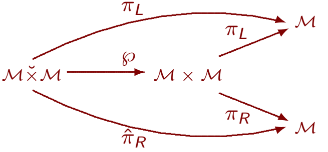
$s = \tanh(t/2) \implies D_t = \frac{1}{2}(1 - s^2)D_s$

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : H_b^s(\mathcal{M}; E) \to H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M}\breve{\times}\mathcal{M}$ supported near $\breve{\Delta}$ s.t.



$$(*) \quad \breve\pi_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $f\!f = \{r = 0\}$.

$[0,\infty \times [-1,1] \ni (r,s) \mapsto (x,x')$ $= \left(\frac{r(1+s)}{2}, \frac{r(1-s)}{2}\right)$

$r = \frac{x+x'}{2}$, $s = \frac{x-x'}{x+x'}$

Try

$\breve\pi_L^* P = \sum_{k+|\alpha|\le m} a_{k,\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

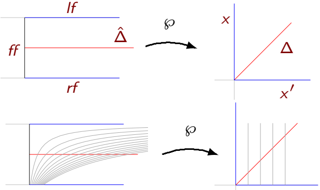
$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \dots, \sigma, \eta)\, d\sigma\, d\eta$$

Without much work, get $(*)$ with smooth $\breve{K}_R$.

Next, find $\breve{K}_{Q_1}$ such that $\breve\pi_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

$$\widehat{\breve\pi_L^* P}|_{f\!f} = \sum_{k+|\alpha|\le m} a_{k,\alpha}(0,y)\left(\sigma\right)^k D_y^\alpha$$

$= \widehat{P}(\sigma)$, invertible if $\mathrm{Re}\,\sigma = 0$.



$s = \tanh(t/2) \implies D_t = \frac{1}{2}(1-s^2)D_s$

Suppose $P \in \mathrm{Diff}_b^m$ is b-elliptic and $0 \notin \mathrm{Im\,spec}_b(P)$. To prove

Suppose $\nu = 0$.

$$P : x^\nu H_b^s(\mathcal{M}; E) \to x^\nu H_b^s(\mathcal{M}; F) \text{ is Fredholm}$$

we construct Q such that $PQ = I - R$ with R compact. Passing to Schwartz kernels and blow up we first find $\breve{K}_Q$ defined on $\mathcal{M} \breve{\times} \mathcal{M}$ supported near $\breve{\Delta}$ s.t.

$$(*) \quad \breve{\pi}_L^* P \breve{K}_Q = \delta_{\breve{\Delta}} - \breve{K}_R$$

with $\breve{K}_R$ smooth across $ff = \{r = 0\}$.

$\breve{K}_R + \breve{\pi}_L^* P \breve{K}_{Q_1} = 0$ on $r = 0$

$\implies$ $\mathrm{op}(\breve{K}_R + \breve{\pi}_L^* P \breve{K}_{Q_1})$ compact:

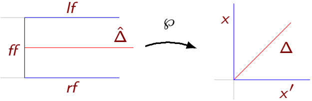
$P(Q + Q_1) = \mathrm{I} - R'$, R' compact

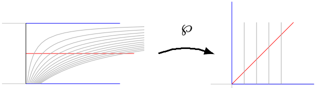
Try

$\breve{\pi}_L^* P = \sum_{k+|\alpha| \le m} a_{k,\alpha}((1+s)r, y)\left(\tfrac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha$

$$\breve{K}_Q = \int e^{is\sigma + i(y-y')\cdot\eta} q(r, s, \ldots, \sigma, \eta)\, d\sigma\, d\eta$$

$r = \dfrac{x + x'}{2}$, $s = \dfrac{x - x'}{x + x'}$





$s = \tanh(t/2) \implies D_t = \frac{1}{2}(1 - s^2) D_s$

Without much work, get $(*)$ with smooth $\breve{K}_R$.

Next, find $\breve{K}_{Q_1}$ such that $\breve{\pi}_L^* P \breve{K}_{Q_1}|_{r=0} = \breve{K}_R|_{r=0}$:

$$\widehat{\breve{\pi}_L^* P}|_{ff} = \sum_{k+|\alpha| \le m} a_{k,\alpha}(0, y)\left(\sigma\right)^k D_y^\alpha$$

$= \widehat{P}(\sigma)$, invertible if $\mathrm{Re}\,\sigma = 0$.

$$\breve{\pi}_L^* P(\breve{K}_Q + \breve{K}_{Q_1}) = \delta_{\breve{\Delta}} - \breve{K}_R + \breve{\pi}_L^* P \breve{K}_{Q_1}$$

Brief explanation, by example:

The operator

$$\Lambda u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \frac{u(x')}{1+|\xi|^2}\, dx'\, d\xi$$

maps $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ but is not compact as an operator

$$L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Brief explanation, by example:

The operator

$$\Lambda u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \frac{u(x')}{1+|\xi|^2}\, dx'\, d\xi$$

maps $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ but is not compact as an operator

$$L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

But the operator

$$u \mapsto (1+|x|^2)^{-\varepsilon}\Lambda u$$

is compact as an operator

$$L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

for any $\varepsilon > 0$.

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\operatorname{Diff}_b^m(\mathcal{M};E,F)$*:*

$$A = x^{-m}P, \quad P \in \operatorname{Diff}_b^m(\mathcal{M};E,F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Assume $m > 0$.

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E,F)$*:*

$$A = x^{-m}P, \quad P \in \mathrm{Diff}_b^m(\mathcal{M};E,F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C_c^\infty(\mathring{\mathcal{M}};E) \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\,\mathrm{Diff}_b^m(\mathcal{M}; E, F)$*:*

$$A = x^{-m} P, \quad P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C_c^\infty(\mathring{\mathcal{M}}; E) \subset x^\gamma L_b^2(\mathcal{M}; E) \to x^\gamma L_b^2(\mathcal{M}; F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\min}(A)$.

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E,F)$*:*

$$A = x^{-m}P, \quad P \in \mathrm{Diff}_b^m(\mathcal{M};E,F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C_c^\infty(\mathring{\mathcal{M}};E) \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\min}(A)$. The space

$$\mathcal{D}_{\max}(A) = \{u \in x^\gamma L_b^2 : Au \in x^\gamma L_b^2\}$$

is the "maximal domain."

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\operatorname{Diff}^m_b(\mathcal{M}; E, F)$*:*

$$A = x^{-m}P, \quad P \in \operatorname{Diff}^m_b(\mathcal{M}; E, F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C^\infty_c(\mathring{\mathcal{M}}; E) \subset x^\gamma L^2_b(\mathcal{M}; E) \to x^\gamma L^2_b(\mathcal{M}; F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\min}(A)$. The space

$$\mathcal{D}_{\max}(A) = \{u \in x^\gamma L^2_b : Au \in x^\gamma L^2_b\}$$

is the "maximal domain." It is a Hilbert space with inner product

$$(u, v)_A = (Au, Av)_{x^\gamma L^2_b} + (u, v)_{x^\gamma L^2_b}$$

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E,F)$:

$$A = x^{-m}P, \quad P \in \mathrm{Diff}_b^m(\mathcal{M};E,F)$$

A is c-elliptic if $x^m A$ is b-elliptic.

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C_c^\infty(\mathring{\mathcal{M}};E) \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\min}(A)$. The space

$$\mathcal{D}_{\max}(A) = \{u \in x^\gamma L_b^2 : Au \in x^\gamma L_b^2\}$$

is the "maximal domain." It is a Hilbert space with inner product

$$(u,v)_A = (Au, Av)_{x^\gamma L_b^2} + (u,v)_{x^\gamma L_b^2}$$

The closure of $C_c^\infty \subset \mathcal{D}_{\max}(A)$ is $\mathcal{D}_{\min}(A)$.

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an elment of $x^{-m}\,\mathrm{Diff}_b^m(\mathcal{M};E,F)$*:*

$$A = x^{-m}P, \quad P \in \mathrm{Diff}_b^m(\mathcal{M};E,F)$$

A is c-elliptic if $x^m A$ *is b-elliptic.*

Fix $\gamma \in \mathbb{R}$, view A as an unbounded operator

Assume $m > 0$.

$$A : C_c^\infty(\mathring{\mathcal{M}};E) \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\min}(A)$. The space

$$\mathcal{D}_{\max}(A) = \{u \in x^\gamma L_b^2 : Au \in x^\gamma L_b^2\}$$

is the "maximal domain." It is a Hilbert space with inner product

$$(u,v)_A = (Au, Av)_{x^\gamma L_b^2} + (u,v)_{x^\gamma L_b^2}$$

The closure of $C_c^\infty \subset \mathcal{D}_{\max}(A)$ is $\mathcal{D}_{\min}(A)$.

Both

$$A : \mathcal{D}_{\min} \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

$$A : \mathcal{D}_{\max} \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};F)$$

are closed, densely defined

Suppose $A \in x^{-m} \operatorname{Diff}_b^m$ is c-elliptic.

Suppose $A \in x^{-m}\,\mathrm{Diff}^m_b$ is c-elliptic.

– *The space* $x^{\gamma+m}H^m_b(\mathcal{M};E)$ *is a subspace of* $\mathcal{D}_{\min}$*, and the inclusion is continuous.*

Suppose $A \in x^{-m}\,\mathrm{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m}H_b^m(\mathcal{M};E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C_c^\infty(\mathbb{R})$ be $=1$ near 0, $\omega_k(x)=\omega(kx)$. If $u \in x^{\gamma+m}H_b^m(\mathcal{M};E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H_b^m(\mathcal{M};E)$. Hence $u_k \to u$ in $x^{\gamma+m}L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $Au_k \to Au$ in $x^\gamma L_b^2$.*

Suppose $A \in x^{-m}\operatorname{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m}H_b^m(\mathcal{M};E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C_c^\infty(\mathbb{R})$ be $=1$ near 0, $\omega_k(x)=\omega(kx)$. If $u \in x^{\gamma+m}H_b^m(\mathcal{M};E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H_b^m(\mathcal{M};E)$. Hence $u_k \to u$ in $x^{\gamma+m}L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $Au_k \to Au$ in $x^\gamma L_b^2$.*

– *$x^{\gamma+m}H_b^m(\mathcal{M};E) = \mathcal{D}_{\min}$ iff $-\gamma - m \notin \operatorname{Im}\operatorname{spec}_b(x^m A)$.*

Suppose $A \in x^{-m} \operatorname{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m} H_b^m(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* Let $\omega \in C_c^\infty(\mathbb{R})$ be $= 1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m} H_b^m(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma+m} H_b^m(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma+m} L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $Au_k \to Au$ in $x^\gamma L_b^2$.

– *$x^{\gamma+m} H_b^m(\mathcal{M}; E) = \mathcal{D}_{\min}$ iff $-\gamma - m \notin \operatorname{Im} \operatorname{spec}_b(x^m A)$.*

If $-\gamma - m \in \operatorname{Im} \operatorname{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Suppose $A \in x^{-m}\,\mathrm{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m}H_b^m(\mathcal{M};E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C_c^\infty(\mathbb{R})$ be $=1$ near 0, $\omega_k(x)=\omega(kx)$. If $u \in x^{\gamma+m}H_b^m(\mathcal{M};E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H_b^m(\mathcal{M};E)$. Hence $u_k \to u$ in $x^{\gamma+m}L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $Au_k \to Au$ in $x^\gamma L_b^2$.*

– *$x^{\gamma+m}H_b^m(\mathcal{M};E)=\mathcal{D}_{\min}$ iff $-\gamma-m \notin \mathrm{Im}\,\mathrm{spec}_b(x^mA)$.*

If $-\gamma-m \in \mathrm{Im}\,\mathrm{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Whatever γ, both

$$A:\mathcal{D}_{\min}\subset x^\gamma L_b^2 \to x^\gamma L_b^2 \quad \text{and} \quad A:\mathcal{D}_{\max}\subset x^\gamma L_b^2 \to x^\gamma L_b^2$$

are Fredholm operators.

Suppose $A \in x^{-m}\,\mathrm{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m}H_b^m(\mathcal{M};E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C_c^\infty(\mathbb{R})$ be $= 1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m}H_b^m(\mathcal{M};E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H_b^m(\mathcal{M};E)$. Hence $u_k \to u$ in $x^{\gamma+m}L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $Au_k \to Au$ in $x^\gamma L_b^2$.*

– $x^{\gamma+m}H_b^m(\mathcal{M};E) = \mathcal{D}_{\min}$ *iff* $-\gamma - m \notin \operatorname{Im}\mathrm{spec}_b(x^m A)$.

If $-\gamma - m \in \operatorname{Im}\mathrm{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Whatever γ, both

$$A : \mathcal{D}_{\min} \subset x^\gamma L_b^2 \to x^\gamma L_b^2 \quad \text{and} \quad A : \mathcal{D}_{\max} \subset x^\gamma L_b^2 \to x^\gamma L_b^2$$

are Fredholm operators.

$A_{\mathcal{D}}$ has finite dimensional kernel and closed range of finite codimension. $\mathrm{Ind}\, A_{\mathcal{D}} = \dim\ker A_{\mathcal{D}} - \dim\operatorname{coker} A_{\mathcal{D}}$.

Suppose $A \in x^{-m}\,\mathrm{Diff}_b^m$ is c-elliptic.

– *The space* $x^{\gamma+m} H_b^m(\mathcal{M}; E)$ *is a subspace of* $\mathcal{D}_{\min}$, *and the inclusion is continuous.* *Let* $\omega \in C_c^\infty(\mathbb{R})$ *be* $= 1$ *near* 0, $\omega_k(x) = \omega(kx)$. *If* $u \in x^{\gamma+m} H_b^m(\mathcal{M}; E)$ *then* $\omega_k u \to u$ *in* $x^{\gamma+m} H_b^m(\mathcal{M}; E)$. *Hence* $u_k \to u$ *in* $x^{\gamma+m} L_b^2$ ($\therefore$ *in* $x^\gamma L_b^2$) *and* $Au_k \to Au$ *in* $x^\gamma L_b^2$.

– $x^{\gamma+m} H_b^m(\mathcal{M}; E) = \mathcal{D}_{\min}$ *iff* $-\gamma - m \notin \mathrm{Im}\,\mathrm{spec}_b(x^m A)$.

If $-\gamma - m \in \mathrm{Im}\,\mathrm{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Whatever γ, both

$$A : \mathcal{D}_{\min} \subset x^\gamma L_b^2 \to x^\gamma L_b^2 \quad \text{and} \quad A : \mathcal{D}_{\max} \subset x^\gamma L_b^2 \to x^\gamma L_b^2$$

are Fredholm operators. The inclusion $\mathcal{D}_{\max} \hookrightarrow x^\gamma L_b^2$ is compact.

$A_{\mathcal{D}}$ has finite dimensional kernel and closed range of finite codimension. $\mathrm{Ind}\, A_{\mathcal{D}} = \dim\ker A_{\mathcal{D}} - \dim\mathrm{coker}\, A_{\mathcal{D}}$.

Suppose $A \in x^{-m}\,\mathrm{Diff}^m_b$ is c-elliptic.

– *The space $x^{\gamma+m}H^m_b(\mathcal{M};E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C^\infty_c(\mathbb{R})$ be $= 1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m}H^m_b(\mathcal{M};E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H^m_b(\mathcal{M};E)$. Hence $u_k \to u$ in $x^{\gamma+m}L^2_b$ ($\therefore$ in $x^\gamma L^2_b$) and $Au_k \to Au$ in $x^\gamma L^2_b$.*

– *$x^{\gamma+m}H^m_b(\mathcal{M};E) = \mathcal{D}_{\min}$ iff $-\gamma - m \notin \mathrm{Im}\,\mathrm{spec}_b(x^m A)$.*

If $-\gamma - m \in \mathrm{Im}\,\mathrm{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Whatever γ, both

$$A : \mathcal{D}_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b \quad \text{and} \quad A : \mathcal{D}_{\max} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$

are Fredholm operators. The inclusion $\mathcal{D}_{\max} \hookrightarrow x^\gamma L^2_b$ is compact.

$A_{\mathcal{D}}$ has finite dimensional kernel and closed range of finite codimension. $\mathrm{Ind}\,A_{\mathcal{D}} = \dim\ker A_{\mathcal{D}} - \dim\operatorname{coker} A_{\mathcal{D}}$.

$A_{\mathcal{D}} - \lambda I$ is a compact perturbation of $A_{\mathcal{D}}$ so $\mathrm{Ind}(A_{\mathcal{D}} - \lambda I) = \mathrm{Ind}\,A_{\mathcal{D}}$.

Suppose $A \in x^{-m} \operatorname{Diff}_b^m$ is c-elliptic.

– *The space $x^{\gamma+m} H_b^m(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\min}$, and the inclusion is continuous.* *Let $\omega \in C_c^\infty(\mathbb{R})$ be $= 1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m} H_b^m(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma+m} H_b^m(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma+m} L_b^2$ ($\therefore$ in $x^\gamma L_b^2$) and $A u_k \to A u$ in $x^\gamma L_b^2$.*

– *$x^{\gamma+m} H_b^m(\mathcal{M}; E) = \mathcal{D}_{\min}$ iff $-\gamma - m \notin \operatorname{Im} \operatorname{spec}_b(x^m A)$.*

If $-\gamma - m \in \operatorname{Im} \operatorname{spec}_b(A)$, then $\mathcal{D}_{\min}$ is more complicated.

Whatever γ, both

$$A : \mathcal{D}_{\min} \subset x^\gamma L_b^2 \to x^\gamma L_b^2 \quad \text{and} \quad A : \mathcal{D}_{\max} \subset x^\gamma L_b^2 \to x^\gamma L_b^2$$

are Fredholm operators. The inclusion $\mathcal{D}_{\max} \hookrightarrow x^\gamma L_b^2$ is compact.

$A_{\mathcal{D}}$ has finite dimensional kernel and closed range of finite codimension. $\operatorname{Ind} A_{\mathcal{D}} = \dim \ker A_{\mathcal{D}} - \dim \operatorname{coker} A_{\mathcal{D}}$.

$A_{\mathcal{D}} - \lambda I$ is a compact perturbation of $A_{\mathcal{D}}$ so $\operatorname{Ind}(A_{\mathcal{D}} - \lambda I) = \operatorname{Ind} A_{\mathcal{D}}$.

This is due to Lesch. A consequence is that $\mathcal{D}_{\min}$ has finite codimension in $\mathcal{D}_{\max}$. Also:

Every extension of $A : \mathcal{D}_{\min} \subset x^\gamma L_b^2 \to x^\gamma L_b^2$ is closed.

Any extension has domain

$$\mathcal{D} = \mathcal{D}_{\min} + D, \qquad D \subset \mathcal{D}_{\max} \text{ finite dimensional.}$$

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u,A^\star v)_{x^\nu L^2_b} \quad \forall u \in C^\infty_c(\mathring{\mathcal{M}};E),\ v \in C^\infty_c(\mathring{\mathcal{M}};F).$$

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u,A^\star v)_{x^\nu L^2_b} \quad \forall u \in C_c^\infty(\mathring{\mathcal{M}};E),\ v \in C_c^\infty(\mathring{\mathcal{M}};F).$$

– The Hilbert space adjoint of A with domain $\mathcal{D}_{\min}(A)$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$.

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u, A^\star v)_{x^\nu L^2_b} \quad \forall u \in C_c^\infty(\mathring{\mathcal{M}};E),\ v \in C_c^\infty(\mathring{\mathcal{M}};F).$$

– *The Hilbert space adjoint of A with domain $\mathcal{D}_{\min}(A)$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$.*
– *The Hilbert space adjoint of A with domain $\mathcal{D}_{\max}(A)$ is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star)$.*

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u,A^\star v)_{x^\nu L^2_b} \quad \forall u \in C^\infty_c(\mathring{\mathcal{M}};E),\ v \in C^\infty_c(\mathring{\mathcal{M}};F).$$

– The Hilbert space adjoint of A with domain $\mathcal{D}_{\min}(A)$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$.
– The Hilbert space adjoint of A with domain $\mathcal{D}_{\max}(A)$ is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star)$.
– The form $[u,v]_A = (Au,v) - (u,A^\star v)$, $u \in \mathcal{D}_{\max}(A),\ v \in \mathcal{D}_{\max}(A^\star)$ is n.d. on $\mathcal{E}(A) \times \mathcal{E}(A^\star)$ and vanishes if $u \in \mathcal{D}_{\min}(A)$ or $v \in \mathcal{D}_{\min}(A^\star)$.

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u,A^\star v)_{x^\nu L^2_b} \quad \forall u\in C_c^\infty(\mathring{\mathcal{M}};E),\ v\in C_c^\infty(\mathring{\mathcal{M}};F).$$

– The Hilbert space adjoint of A with domain $\mathcal{D}_{\min}(A)$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$.
– The Hilbert space adjoint of A with domain $\mathcal{D}_{\max}(A)$ is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star)$.
– The form $[u,v]_A = (Au,v)-(u,A^\star v)$, $u\in\mathcal{D}_{\max}(A),\ v\in\mathcal{D}_{\max}(A^\star)$ is n.d. on $\mathcal{E}(A)\times\mathcal{E}(A^\star)$ and vanishes if $u\in\mathcal{D}_{\min}(A)$ or $v\in\mathcal{D}_{\min}(A^\star)$.
– The orthogonal complement of $\mathcal{D}_{\min}(A)$ in $\mathcal{D}_{\max}(A)$ is

$$\mathcal{E}(A) = \ker(A^\star A+\mathrm{I})\cap\mathcal{D}_{\max}(A)$$

This is a finite-dimensional space.

Adjoints

Let $A^\star$ denote the *formal* adjoint of A on $x^\nu L^2_b(\mathcal{M};E)$:

$$(Au,v)_{x^\nu L^2_b} = (u,A^\star v)_{x^\nu L^2_b} \quad \forall u\in C_c^\infty(\mathring{\mathcal{M}};E),\ v\in C_c^\infty(\mathring{\mathcal{M}};F).$$

– The Hilbert space adjoint of A with domain $\mathcal{D}_{\min}(A)$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$.
– The Hilbert space adjoint of A with domain $\mathcal{D}_{\max}(A)$ is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star)$.
– The form $[u,v]_A = (Au,v)-(u,A^\star v)$, $u\in\mathcal{D}_{\max}(A),\ v\in\mathcal{D}_{\max}(A^\star)$ is n.d. on $\mathcal{E}(A)\times\mathcal{E}(A^\star)$ and vanishes if $u\in\mathcal{D}_{\min}(A)$ or $v\in\mathcal{D}_{\min}(A^\star)$.
– The orthogonal complement of $\mathcal{D}_{\min}(A)$ in $\mathcal{D}_{\max}(A)$ is

$$\mathcal{E}(A) = \ker(A^\star A+\mathrm{I})\cap\mathcal{D}_{\max}(A)$$

This is a finite-dimensional space.
– if $u\in\mathcal{E}(A)$, then $Au\in\mathcal{E}(A^\star)$.

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\mathsf{max}}(A)$:

$$\begin{aligned} u \in \mathcal{D}_{\mathsf{min}}^\perp &\iff \forall v \in \mathcal{D}_{\mathsf{min}} : (u, v)_A = 0 \\ &\iff \forall v \in \mathcal{D}_{\mathsf{min}} : (Au, Av) + (u, v) = 0 \\ &\iff \forall v \in \mathcal{D}_{\mathsf{min}} : (Au, Av) = -(u, v) \end{aligned}$$

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$u \in \mathcal{D}_{\min}^{\perp} \iff \forall v \in \mathcal{D}_{\min} : (u, v)_A = 0$$

$$\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u, v) = 0$$

$$\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u, v)$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$\begin{aligned} u \in \mathcal{D}_{\min}^{\perp} &\iff \forall v \in \mathcal{D}_{\min} : (u, v)_A = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u, v) = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u, v) \\ &\iff Au \in \mathcal{D}_{\max}(A^\star) \text{ and } A^\star Au = -u. \end{aligned}$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$
\begin{aligned}
u \in \mathcal{D}_{\min}^\perp &\iff \forall v \in \mathcal{D}_{\min} : (u, v)_A = 0 \\
&\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u, v) = 0 \\
&\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u, v) \\
&\iff Au \in \mathcal{D}_{\max}(A^\star) \text{ and } A^\star Au = -u.
\end{aligned}
$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^\star)$:

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$\begin{aligned} u \in \mathcal{D}_{\min}^{\perp} &\iff \forall v \in \mathcal{D}_{\min} : (u, v)_A = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u, v) = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u, v) \\ &\iff Au \in \mathcal{D}_{\max}(A^\star) \text{ and } A^\star Au = -u. \end{aligned}$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^\star)$:
The above says $Au \in \mathcal{D}_{\max}(A^\star)$ and $A^\star Au + u = 0$.

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$\begin{aligned} u \in \mathcal{D}_{\min}^{\perp} &\iff \forall v \in \mathcal{D}_{\min} : (u, v)_A = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u, v) = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u, v) \\ &\iff Au \in \mathcal{D}_{\max}(A^\star) \text{ and } A^\star Au = -u. \end{aligned}$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^\star)$:
The above says $Au \in \mathcal{D}_{\max}(A^\star)$ and $A^\star Au + u = 0$.
Applying A:

$$A(A^\star A)u + Au = 0$$

Proof of $\mathcal{E}(A) = \ker(A^\star A + \mathrm{I}) \cap \mathcal{D}_{\max}(A)$:

$$\begin{aligned} u \in \mathcal{D}_{\min}^\perp &\iff \forall v \in \mathcal{D}_{\min} : (u,v)_A = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) + (u,v) = 0 \\ &\iff \forall v \in \mathcal{D}_{\min} : (Au, Av) = -(u,v) \\ &\iff Au \in \mathcal{D}_{\max}(A^\star) \text{ and } A^\star A u = -u. \end{aligned}$$

$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $v \mapsto (-u, v)$

In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^\star)$:
The above says $Au \in \mathcal{D}_{\max}(A^\star)$ and $A^\star A u + u = 0$.
Applying A:

$$A(A^\star A)u + Au = 0$$

that is

$$(AA^\star)Au + Au = 0$$

so $Au \in \mathcal{E}(A^\star)$ by what we just proved.

The elements of $\mathcal{E}(A)$ in some sense live on the boundary of $\mathcal{M}$.

Namely, let

$$\pi_{\max}, \pi_{\min} : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$$

be the orthogonal projections on, respectively, $\mathcal{E}(A)$ and $\mathcal{D}_{\min}$. $\pi_{\max} = \mathrm{I} - \pi_{\min}$

Since A is elliptic in $\mathring{\mathcal{M}}$ in the usual sense, so is $A^\star A + \mathrm{I}$. If $u \in \mathcal{E}$ then $(A^\star A + \mathrm{I})u = 0$, so u is smooth in $\mathring{\mathcal{M}}$.

Let ω be a smooth function on $\mathcal{M}$, $= 1$ near $\partial\mathcal{M}$. If $u \in \mathcal{E}(A)$ then $\omega u \in \mathcal{D}_{\max}$ because

$$\omega u = u - (1-\omega)u$$

and $(1-\omega)u \in C_c^\infty \subset \mathcal{D}_{\min}$. And so $\pi_{\max}((1-\omega)u) = 0$

$$\pi_{\max}(\omega u) = \pi_{\max}(u) - \pi_{\max}((1-\omega)u) = \pi_{\max} u = u.$$

It is in this sense that u "lives" on the boundary.

Closed extensions, cont.

Any extension of $A : \mathcal{D}_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b$ has as domain a space

$$\mathcal{D} = \mathcal{D}_{\min} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

Closed extensions, cont.

Any extension of $A : \mathcal{D}_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b$ has as domain a space

$$\mathcal{D} = \mathcal{D}_{\min} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

(Lesch) *The index of*

$$A : \mathcal{D} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$

is

$$\operatorname{Ind} A_{\mathcal{D}} = \operatorname{Ind} A_{\mathcal{D}_{\min}} + \dim D.$$

Closed extensions, cont.

Any extension of $A : \mathcal{D}_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b$ has as domain a space

$$\mathcal{D} = \mathcal{D}_{\min} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

(Lesch) *The index of*

$$A : \mathcal{D} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$

is

$$\operatorname{Ind} A_{\mathcal{D}} = \operatorname{Ind} A_{\mathcal{D}_{\min}} + \dim D.$$

So the closed extensions of A of a given index are in one to one correspondence with the subspaces of $\mathcal{E}$ of a given dimension.

Closed extensions, cont.

Any extension of $A : \mathcal{D}_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b$ has as domain a space

$$\mathcal{D} = \mathcal{D}_{\min} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

(Lesch) *The index of*

$$A : \mathcal{D} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$

is

$$\operatorname{Ind} A_{\mathcal{D}} = \operatorname{Ind} A_{\mathcal{D}_{\min}} + \dim D.$$

So the closed extensions of A of a given index are in one to one correspondence with the subspaces of $\mathcal{E}$ of a given dimension. This is:

The set of closed extensions of A of index $\operatorname{Ind} A_{\mathcal{D}_{\min}} + k$ is parametrized by the Grasmannian $\mathrm{Gr}_k(\mathcal{E})$ of k-dimensional subspaces of $\mathcal{E}$.

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_{\mathcal{D}}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star}$$

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_\mathcal{D}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star} = (A^\star Au, A^\star v) + (Au, v)$$

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_{\mathcal{D}}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star} = (A^\star Au, A^\star v) + (Au, v) \;= -(u, A^\star v) + (Au, v)$$

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_{\mathcal{D}}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star} = (A^\star Au, A^\star v) + (Au, v) = -(u, A^\star v) + (Au, v)$$

So $D \ni u \mapsto (Au, v)$ is continuous $(= (A^\star v, u))$. Since

$$(v, Au) = (A^\star v, u) \text{ if } u \in \mathcal{D}_{\min},$$

$v \in D'$.

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_{\mathcal{D}}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star} = (A^\star Au, A^\star v) + (Au, v) = -(u, A^\star v) + (Au, v)$$

So $D \ni u \mapsto (Au, v)$ is continuous $(= (A^\star v, u))$. Since

$$(v, Au) = (A^\star v, u) \text{ if } u \in \mathcal{D}_{\min},$$

$v \in D'$.

Conversely, if $v \in D'$, then $(Au, v) = (u, A^\star v)$ if $u \in D$, i.e.

Adjoints, cont.

The Hilbert space adjoint of A with domain $\mathcal{D} = \mathcal{D}_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^\star$ with domain $\mathcal{D}_{\min}(A^\star) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\max}(A^\star)$.

Proof: Let the domain of the adjoint of $A_{\mathcal{D}}$ be $\mathcal{D}_{\min}(A^\star) + D'$, $D' \subset \mathcal{E}(A^\star)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^\star} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^\star} = (A^\star Au, A^\star v) + (Au, v) \ = -(u, A^\star v) + (Au, v)$$

So $D \ni u \mapsto (Au, v)$ is continuous ($= (A^\star v, u)$). Since

$$(v, Au) = (A^\star v, u) \text{ if } u \in \mathcal{D}_{\min},$$

$v \in D'$.

Conversely, if $v \in D'$, then $(Au, v) = (u, A^\star v)$ if $u \in D$, i.e.

$$0 = (Au, v) - (u, A^\star v) = (Au, v) + (A^\star Au, A^\star v) = (Au, v)_{A^\star}$$

Let $d = \dim \mathcal{E}(A)$, $d^\star = \dim \mathcal{E}(A^\star)$. Then

$$\mathcal{J} : \mathrm{Gr}_k(\mathcal{E}(A)) \to \mathrm{Gr}_{d^\star - k}(\mathcal{E}(A^\star)), \quad \mathcal{J} D = A(D)^\perp$$

This is a continuous (holomorphic) map.

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\mathsf{min}} + D$ has nonempty resolvent set:

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\mathsf{min}} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\mathsf{min}} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\mathrm{Ind}(A_{\mathcal{D}}) = 0$.

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\min} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\min}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\min}} \le 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L^2_b$ is compact.

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\min} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\min}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\min}} \leq 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \leq \operatorname{Ind} A_{\mathcal{D}_{\max}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\max}} \geq 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L^2_b$ is compact.

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\text{min}}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\text{min}}} \leq 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \leq \operatorname{Ind} A_{\mathcal{D}_{\text{max}}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\text{max}}} \geq 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\text{max}} \hookrightarrow x^{\gamma} L^2_b$ is compact.

A has an extension with nonempty resolvent set iff

$$\operatorname{Ind} A_{\mathcal{D}_{\text{min}}} \leq 0 \quad \text{and} \quad \operatorname{Ind} A_{\mathcal{D}_{\text{max}}} \geq 0.$$

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\min} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\min}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\min}} \leq 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \leq \operatorname{Ind} A_{\mathcal{D}_{\max}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\max}} \geq 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L^2_b$ is compact.

A has an extension with nonempty resolvent set iff

$$\operatorname{Ind} A_{\mathcal{D}_{\min}} \leq 0 \quad \text{and} \quad \operatorname{Ind} A_{\mathcal{D}_{\max}} \geq 0.$$

If $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is invertible, then it is injective,

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\min} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\min}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\min}} \le 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \le \operatorname{Ind} A_{\mathcal{D}_{\max}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\max}} \ge 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L^2_b$ is compact.

A has an extension with nonempty resolvent set iff

$$\operatorname{Ind} A_{\mathcal{D}_{\min}} \le 0 \quad \text{and} \quad \operatorname{Ind} A_{\mathcal{D}_{\max}} \ge 0.$$

If $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is invertible, then it is injective, so also

$$A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \text{ is injective.}$$

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\min} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\min}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\min}} \le 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \le \operatorname{Ind} A_{\mathcal{D}_{\max}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\max}} \ge 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L^2_b$ is compact.

A has an extension with nonempty resolvent set iff

$$\operatorname{Ind} A_{\mathcal{D}_{\min}} \le 0 \quad \text{and} \quad \operatorname{Ind} A_{\mathcal{D}_{\max}} \ge 0.$$

If $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is invertible, then it is injective, so also

$$A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \text{ is injective.}$$

Also $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is surjective,

Spectra

Suppose A with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

$$A_{\mathcal{D}} - \lambda_0 \mathrm{I} \text{ is invertible for some } \lambda_0$$

Then $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \operatorname{Ind} A_{\mathcal{D}_{\text{min}}}$, hence $\operatorname{Ind} A_{\mathcal{D}_{\text{min}}} \le 0$. Also $\operatorname{Ind} A_{\mathcal{D}} \le \operatorname{Ind} A_{\mathcal{D}_{\text{max}}}$, so $\operatorname{Ind} A_{\mathcal{D}_{\text{max}}} \ge 0$.

$\operatorname{Ind}(A_{\mathcal{D}} - \lambda_0 \mathrm{I}) = \operatorname{Ind}(A_{\mathcal{D}})$ because $\mathcal{D}_{\text{max}} \hookrightarrow x^{\gamma} L^2_b$ is compact.

A has an extension with nonempty resolvent set iff

$$\operatorname{Ind} A_{\mathcal{D}_{\text{min}}} \le 0 \quad \text{and} \quad \operatorname{Ind} A_{\mathcal{D}_{\text{max}}} \ge 0.$$

If $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is invertible, then it is injective, so also

$$A_{\mathcal{D}_{\text{min}}} - \lambda_0 \mathrm{I} \text{ is injective.}$$

Also $A_{\mathcal{D}} - \lambda_0 \mathrm{I}$ is surjective, so

$$A_{\mathcal{D}_{\text{max}}} - \lambda_0 \mathrm{I} \text{ is surjective.}$$

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \textit{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 \mathrm{I} \textit{ is surjective}\}$$

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \textit{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 \mathrm{I} \textit{ is surjective}\}$$

The background spectrum is the complement of this:

$$\text{bg-spec}(A) = \mathbb{C} \setminus \text{bg-res}(A).$$

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \textit{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 \mathrm{I} \textit{ is surjective}\}$$

The background spectrum is the complement of this:

$$\text{bg-spec}(A) = \mathbb{C} \setminus \text{bg-res}(A).$$

Relevancy:

1. The background spectrum of A is contained in the spectrum of any extension of A.

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 I \textit{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 I \textit{ is surjective}\}$$

The background spectrum is the complement of this:

$$\text{bg-spec}(A) = \mathbb{C} \setminus \text{bg-res}(A).$$

Relevancy:

1. The background spectrum of A is contained in the spectrum of any extension of A.

2. For $\lambda \in \text{bg-res}(A)$ *let*

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda I).$$

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \text{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 \mathrm{I} \text{ is surjective}\}$$

The background spectrum is the complement of this:

$$\text{bg-spec}(A) = \mathbb{C} \setminus \text{bg-res}(A).$$

Relevancy:

1. The background spectrum of A is contained in the spectrum of any extension of A.

2. For $\lambda \in \text{bg-res}(A)$ *let*

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}).$$

Then

$$\operatorname{spec} A_{\mathcal{D}} = \text{bg-spec}(A) \cup \{\lambda \in \text{bg-res}(A) : \mathcal{K}_\lambda \cap \mathcal{D} \neq 0\}$$

This is a disjoint union.

The background resolvent of A is

$$\text{bg-res}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda_0 \mathrm{I} \textit{ is injective, } A_{\mathcal{D}_{\max}} - \lambda_0 \mathrm{I} \textit{ is surjective}\}$$

The background spectrum is the complement of this:

$$\text{bg-spec}(A) = \mathbb{C} \setminus \text{bg-res}(A).$$

Relevancy:

1. *The background spectrum of A is contained in the spectrum of any extension of A.*

2. *For* $\lambda \in \text{bg-res}(A)$ *let*

$\bigsqcup \mathcal{K}_\lambda \to \text{bg-res}(A)$ is a holomorphic vector bundle.

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}).$$

Then

$$\operatorname{spec} A_{\mathcal{D}} = \text{bg-spec}(A) \cup \{\lambda \in \text{bg-res}(A) : \mathcal{K}_\lambda \cap \mathcal{D} \neq 0\}$$

This is a disjoint union.

The operator $A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}$ has a right inverse if $\lambda \in \text{bg-res}(A)$ because it is surjective. Choose it so that it maps onto $\mathcal{K}_\lambda^\perp$, call it $B_{\max}(\lambda)$:

$$B_{\max}(\lambda) \text{ is the inverse of } (A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) : \mathcal{K}_\lambda^\perp \to x^\gamma L^2_b;$$

it depends smoothly (real-analytically) on λ. $B_{\max}(\lambda)(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) = \pi_{\mathcal{K}_\lambda^\perp}$

The operator $A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}$ has a right inverse if $\lambda \in \text{bg-res}(A)$ because it is surjective. Choose it so that it maps onto $\mathcal{K}_\lambda^\perp$, call it $B_{\max}(\lambda)$:

$$B_{\max}(\lambda) \text{ is the inverse of } (A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) : \mathcal{K}_\lambda^\perp \to x^\gamma L_b^2;$$

it depends smoothly (real-analytically) on λ.

$B_{\max}(\lambda)(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) = \pi_{\mathcal{K}_\lambda^\perp}$

Let $\mathcal{D} = \mathcal{D}_{\min} + D$ be any domain of index 0. If $\mathcal{K}_\lambda \cap \mathcal{D} = 0$, then $\mathcal{D} + \mathcal{K}_\lambda = \mathcal{D}_{\max}$.

$$\pi_{\mathcal{K}_\lambda, \mathcal{D}} = \text{ projection on } \mathcal{K}_\lambda \text{ along } \mathcal{D}.$$

The operator $A_{\mathcal{D}_{\max}} - \lambda I$ has a right inverse if $\lambda \in \text{bg-res}(A)$ because it is surjective. Choose it so that it maps onto $\mathcal{K}_\lambda^\perp$, call it $B_{\max}(\lambda)$:

$$B_{\max}(\lambda) \text{ is the inverse of } (A_{\mathcal{D}_{\max}} - \lambda I) : \mathcal{K}_\lambda^\perp \to x^\gamma L_b^2;$$

it depends smoothly (real-analytically) on λ.

$B_{\max}(\lambda)(A_{\mathcal{D}_{\max}} - \lambda I) = \pi_{\mathcal{K}_\lambda^\perp}$

Let $\mathcal{D} = \mathcal{D}_{\min} + D$ be any domain of index 0. If $\mathcal{K}_\lambda \cap \mathcal{D} = 0$, then $\mathcal{D} + \mathcal{K}_\lambda = \mathcal{D}_{\max}$.

$$\pi_{\mathcal{K}_\lambda,\mathcal{D}} = \text{ projection on } \mathcal{K}_\lambda \text{ along } \mathcal{D}.$$

Then

$$(A - \lambda I)[B_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda)] = I$$

and

$$\begin{aligned}[B_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda)](A - \lambda I) &= (I - \pi_{\mathcal{K}_\lambda,\mathcal{D}}) B_{\max}(\lambda)(A - \lambda I) \\ &= \pi_{\mathcal{D},\mathcal{K}_\lambda} \pi_{\mathcal{K}_\lambda^\perp} = \pi_{\mathcal{D},\mathcal{K}_\lambda}(\pi_{\mathcal{K}_\lambda^\perp} + \pi_{\mathcal{K}_\lambda}) = \pi_{\mathcal{D},\mathcal{K}_\lambda}\end{aligned}$$

So

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda)$$

is the resolvent of $A_{\mathcal{D}} - \lambda I$.

Suppose $-d' = \operatorname{Ind} A_{\mathcal{D}_{\min}} < 0$ *and* $\operatorname{Ind} A_{\mathcal{D}_{\max}} > 0$. *For any* $\lambda \in \mathbb{C}$ *there is* $\mathcal{D} = \mathcal{D}_{\min} + D$ *with* $D \in \operatorname{Gr}_{d'}$ *such that* $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$.

Suppose $-d' = \operatorname{Ind} A_{\mathcal{D}_{\min}} < 0$ *and* $\operatorname{Ind} A_{\mathcal{D}_{\max}} > 0$. *For any* $\lambda \in \mathbb{C}$ *there is* $\mathcal{D} = \mathcal{D}_{\min} + D$ *with* $D \in \mathrm{Gr}_{d'}$ *such that* $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$.

Proof. If $\lambda \in \text{bg-spec}(A)$, then $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$ for any $\mathcal{D}$.

Suppose $-d' = \operatorname{Ind} A_{\mathcal{D}_{\min}} < 0$ *and* $\operatorname{Ind} A_{\mathcal{D}_{\max}} > 0$*. For any* $\lambda \in \mathbb{C}$ *there is* $\mathcal{D} = \mathcal{D}_{\min} + D$ *with* $D \in \operatorname{Gr}_{d'}$ *such that* $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$*.*

Proof. If $\lambda \in$ bg-spec(A), then $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$ for any $\mathcal{D}$. So suppose $\lambda \in$ bg-spec(A). We have $\mathcal{K}_\lambda \cap \mathcal{D}_{\min} = 0$ (otherwise $A_{\mathcal{D}_{\min}} - \lambda \mathrm{I}$ is not injective). Also,

$$d'' = \operatorname{Ind}(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) = \dim \ker(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I}) - \dim \operatorname{coker}(A_{\mathcal{D}_{\max}} - \lambda \mathrm{I})$$

so $\dim \mathcal{K}_\lambda = d'' > 0$. Pick $\phi \in \mathcal{K}_\lambda$, $\phi \neq 0$. Pick a subspace D_0 of $\mathcal{E}$ of dimension $d' - 1$ such that $\phi \notin D' + \mathcal{D}_{\min}$ and let
$\mathcal{D} = \mathcal{D}_{\min} + \operatorname{span} \phi + D' = \mathcal{D}$. Then $\mathcal{D} = \mathcal{D}_{\min} + D$ with $D \in \operatorname{Gr}_{d'}$

J. Gil, T. Krainer, and G. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, Canad. J. Math. 59 (2007), 742-794.

_______, *Resolvents of elliptic cone operators*, J. Funct. Anal. **241** (2006), 1–55.

_______, *On rays of minimal growth for elliptic cone operators*, Oper. Theory Adv. Appl. **172** (2007), 33–50

J. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Amer. J. Math. **125** (2003) 2, 357–408.

M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.

B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, Studies in Mathematics and its Applications, 24. North-Holland Publishing Co., Amsterdam, 1991.