Elliptic operators on manifolds with conical singularities, I

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Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathring{\mathcal{M}}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M}\backslash\mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

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2. while at the same time regarding the functions

$$(-\infty, 0) \times \mathcal{N}_j \ni (t, p) \mapsto x = e^t, \quad j = 1, 2, \ldots$$

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The compactification is diffeomorphic to $\mathcal{K}$.

b-metrics

The cylindrical metric $dt^2 + h_j$ on $(-\infty, 0) \times \mathcal{N}_j$, where h_j is a metric on $\mathcal{N}_j$, is transformed to

$$\frac{dx^2}{x^2} + h_j$$

under the change $t = \log x$.

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Let $\mathcal{M}$ be a compact manifold with boundary, let x be a defining function for its boundary, positive in $\overset{\circ}{\mathcal{M}}$.

A b-metric on a compact manifold $\mathcal{M}$ with boundary together is a smooth Riemannian metric on $\overset{\circ}{\mathcal{M}}$ which is of the form

$$g = a\frac{dx^2}{x^2} + h$$

near $\partial\mathcal{M}$, where h is such smooth and whose restriction to $\{x = \varepsilon\}$ is a Riemannian metric for each $\varepsilon \geq 0$.

b-Laplacian

A b-Laplacian is, naturally, the Laplace operator of a b-metric. Near $\partial\mathcal{M}$ (on functions) it has the general form

$$a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j}\, xD_x\, D_{y_j} + \sum_{i,j=1}^{n} a_{ij} D_{y_i} D_{y_j} + a_0 xD_x + \sum_{j=1}^{n} a_j D_{y_j} \qquad (*)$$

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The operator (*) is constructed using vector the fields $x\partial_x$ and ∂_{y_j}: these are tangential vector fields.

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Let

$$C^\infty_{\mathrm{tan}}(\mathcal{M}; T\mathcal{M}) = \{X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial\mathcal{M}} \text{ is tangent to } \partial\mathcal{M}\}$$

There is

$$\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$$

(a bundle homomorphism)

such that

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$C_{\tan}(\mathcal{M}; T\mathcal{M})$ is a finitely generated projective module over $C(\mathcal{M})$. By a theorem of Swan there is a vector bundle ${}^bT\mathcal{M} \to \mathcal{M}$ whose space of continuous sections is $C_{\tan}(\mathcal{M}; T\mathcal{M})$.

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If $x, y_1, \dots, y_n$ is a local chart of $\mathcal{M}$ near $p \in \partial\mathcal{M}$, then

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ev is a bundle isomorphism over $\mathring{\mathcal{M}}$, has 1-dimensional kernel over $\partial\mathcal{M}$.
$x\partial_x$ is a canonical section of ${}^bT\mathcal{M}$ along $\partial\mathcal{M}$

The b-cotangent bundle; b-metrics

The b-cotangent bundle is the dual of the b-tangent bundle:

$$ {}^bT^*\mathcal{M} = ({}^bT\mathcal{M})^* $$

If $x\partial_x, \partial_{y_1}, \dots, \partial_{y_n}$ is a frame of ${}^bT\mathcal{M}$ near $p_0 \in \partial\mathcal{M}$, then

$$ \frac{dx}{x}, dy_1, \dots, dy_n $$

is a frame of ${}^bT^*\mathcal{M}$ near p_0. The bundle map $\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$ gives

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A b-metric is a Riemannian metric on ${}^bT\mathcal{M}$. Locally such metrics have the form

$$ g_{00}\frac{dx}{x} \otimes \frac{dx}{x} + \sum_j g_{0j}\Big(\frac{dx}{x} \otimes dy_j + dy^j \otimes \frac{dx}{x}\Big) + \sum_{i,j} a_{ij}\, dy_i \otimes dy_j . $$

b-Differential operators

A linear b-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial\mathcal{M}$ has the form

x is a defining function for $\partial\mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$

$$\sum_{\alpha+k\leq m} a_{k\alpha}(xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.$$

More generally:

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a b-differential operator of order m if

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$$\sum_{\alpha+k\le m} a_{k\alpha}(xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.$$

$\operatorname{Diff}_b^m(\mathcal{M}; E, F)$ is the space of linear b-differential operators on $\mathcal{M}$ of order m with smooth coefficients.

More generally:

Let E, $F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a b-differential operator of order m if

$\operatorname{Diff}^m(\mathcal{M}; E, F)$ *is the space of linear differential operators of order m with smooth coefficients.*

$$x^{-\nu} P x^{\nu} \in \operatorname{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m.$$

Example: Let $P = a\partial_x$. Then

$$x^{-1}Pxu = x^{-1}a\partial_x xu = a\partial_x u + x^{-1}au.$$

If this has smooth coefficients, then $a = xa'$. So $P = a'x\partial_x$.

b-Symbol

The vector bundle map $\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathring{\mathcal{M}}$, so the dual map $\mathrm{ev}^* : T^*\mathcal{M} \to {}^bT^*\mathcal{M}$ is also an isomorphism over $\mathring{\mathcal{M}}$.

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\boldsymbol{\sigma}(P) : \pi^* E \to \pi^* F$$

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b-ellipticity

Naturally,

> *The operator $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ is b-elliptic if ${}^b\boldsymbol{\sigma}(P)$ is invertible on ${}^bT^*\mathcal{M}\backslash 0$.*

Example: Let $g = \dfrac{dx^2}{x^2} + dy^2$ on $[0, \infty) \times S^1$. Then

$$\Delta = -(x\partial_x)^2 - \partial_y^2$$

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In general, if $P = \sum_{k+|\alpha|\le m} a_{k\alpha}(xD_x)^k D_y^\alpha$, then

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Mellin transform

If $\mathcal{M}$ is a manifold with boundary, then

$$\dot{C}^\infty(M) = \{u \in C^\infty(\mathcal{M}) : u \text{ vanishes to infinite order on } \partial\mathcal{M}\}$$

Let $u \in \dot{C}^\infty[0,\infty)$ be compactly supported. The Mellin transform of u is

$$\mathscr{M}(u)(\sigma) = \int_0^\infty x^{-i\sigma} u(x)\, \frac{dx}{x}$$

The inverse Mellin transform is

$$\mathscr{M}^{-1}(v) = \frac{1}{2\pi} \int_{\mathrm{Re}\,\sigma=0} x^{i\sigma} v(\sigma)\, d\sigma.$$

The Mellin transform is the Fourier transform with e^t replaced by x.

If $u \in L^2([0,\infty); \dfrac{dx}{x})$ then
$$\mathscr{M}(u) \in L^2(\mathbb{R}).$$

Further, if $u \in L^2([0,\infty); \dfrac{dx}{x})$ has compact support, then $\mathscr{M}(u)$ is holomorphic in $\operatorname{Im}\sigma > 0$: Since $|x^{-i\sigma}| = x^{\operatorname{Im}\sigma}$,
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A (smooth) b-density is a density on $\overset{\circ}{\mathcal{M}}$ of the form $\mathfrak{m}_b = \frac{1}{x}\mathfrak{m}$ where $\mathfrak{m}$ is a smooth positive density on $\mathcal{M}$. Define
$$L^2_b(\mathcal{M}) = L^2(\mathcal{M}, \mathfrak{m}_b)$$

Let $\mathcal{N} = \partial\mathcal{M}$, let $[0,\varepsilon) \times \mathcal{N}$ be a tubular neighborhood of $\mathcal{N}$ in $\mathcal{M}$. Let $\omega \in C^\infty(\mathcal{M})$, $\omega = 1$ near $\mathcal{N}$, ω compactly supported in $[0,\varepsilon) \times \mathcal{N}$.
$$\mathscr{M}(u)(\sigma, p) = \int_0^\infty x^{-i\sigma}\omega(x,p)u(x,p)\,\frac{dx}{x}, \quad p \in \mathcal{N},\ \sigma \in \mathbb{C},\ \operatorname{Im}\sigma \geq 0.$$

If $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$, then P defines an operator P_b on $\mathcal{N} = \partial\mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of u, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$.

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$P\tilde{u}\big|_{\mathcal{N}}$ depends only on u. The difference between two extensions of u is xv, $v \in C^\infty(\mathcal{M}; E)$. By definition of b operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if v is. So $P(xv)\big|_{\mathcal{N}} = 0$.

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$P\tilde{u}\big|_{\mathcal{N}}$ depends only on u. The difference between two extensions of u is xv, $v \in C^\infty(\mathcal{M}; E)$. By definition of b operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if v is. So $P(xv)\big|_{\mathcal{N}} = 0$.

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The indicial operator of $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ is

$$\widehat{P}(\sigma) = (x^{-i\sigma}Px^{i\sigma})_b$$

$\widehat{P}(\sigma)$ is a polynomial in $\sigma \in \mathbb{C}$ with values in $\mathrm{Diff}(\mathcal{N}; E, F)$.

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha|\leq m} a_{k\alpha}(x,y)(xD_x)^k D_y^\alpha$$

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$$\begin{aligned}\sum_{k+|\alpha|=m} a_{k\alpha}(0,y)\sigma^k\eta^\alpha &= \sum_{k+|\alpha|=m} a_{k\alpha}(0,y)(\mathrm{Re}\,\sigma + i\mathrm{Im}\,\sigma)^k\eta^\alpha\\ &= \sum_{k+|\alpha|=m} a_{k\alpha}(0,y)(\mathrm{Re}\,\sigma)^k(1+i\mathrm{Im}\,\sigma/\mathrm{Re}\,\sigma)^k\eta^\alpha\\ &= {}^b\boldsymbol{\sigma}(P)(0,y,\mathrm{Re}\,\sigma,\eta)\big(\mathrm{I}+\mathcal{O}(|\mathrm{Re}\,\sigma|^{-1})\big)\end{aligned}$$

if $|\mathrm{Im}\,\sigma/\mathrm{Re}\,\sigma| < c$ with small enough $c > 0$.

Using the invertibility of

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(with $|\mathrm{Im}\,\sigma| < c|\mathrm{Re}\,\sigma|$) one finds a family of pseudodifferential operators $Q(\sigma)$ on $\mathcal{N}$ such that

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$$\widehat{P}(\sigma) : H^m(\mathcal{N}) \subset L^2(\mathcal{N}) \to L^2(\mathcal{N})$$

is a holomorphic Fredholm family which is invertible at some point, the set

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is discrete without points of accumulation. For any $a > 0$, the set

$\mathrm{spec}_b(P) \cap \{\sigma : |\mathrm{Im}\,\sigma| < a\}$ is finite.

Sobolev spaces

$L^2_b = L^2(\mathcal{M}, \frac{1}{x}\mathfrak{m})$

1. If s is a nonnegative integer, then $H^s_b(\mathcal{M})$ consists of all $u \in L^2_b(\mathcal{M})$ such that

$$X_1 \dots X_k u \in L^2_b(\mathcal{M}) \text{ for all } X_1, \dots, X_k \in C^\infty(\mathcal{M}; {}^bT^*\mathcal{M}), k \le s$$

2. The space $H^{-s}_b(\mathcal{M})$ is the dual of $H^s_b(\mathcal{M})$
3. If s is not an integer, then $H^s_b(\mathcal{M})$ is defined by interpolation.
4. If s and ν are real numbers, then $x^\nu H^s_b(\mathcal{M}) = \{x^\nu u : u \in H^s_b(\mathcal{M})\}$.

If $P \in \mathrm{Diff}^m_b(\mathcal{M}; E, F)$, then $P : x^\nu H^s_b(\mathcal{M}; E) \to x^\nu H^{s-m}_b(\mathcal{M}; F)$ is continuous.

Elliptic regularity

Let $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ *be b-elliptic. If* $u \in x^\nu H_b^s(\mathcal{M}; E)$ *and* $Pu \in x^\nu H_b^s(\mathcal{M}; F)$*, then* $u \in x^\nu H_b^{s+m}(\mathcal{M}; E)$*.*

The proof is by construction of an operator Q such that

$$QP = \mathrm{I} - R$$

with $Q : x^\nu H_b^t \to x^\nu H_b^{t+m}$ and $R : x^\nu H_b^t \to x^\nu H_b^\infty$ for any t.

Parametrices

The operator Q is obtained by constructing its parametrix. Suppose for the moment that

$$P = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$$

is an operator on an open set $U \subset \mathbb{R}^n$. If $u \in C_c^\infty(U)$ then

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Big[\int_U e^{-ix'\cdot\xi} u(x')\, dx' \Big]\, d\xi$$

so

$$Pu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \Big[\int_U e^{-ix'\cdot\xi} u(x')\, dx' \Big]\, d\xi$$

Let $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$. If P is elliptic, that is,

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Suppose $P = p(x, D_x)$ is elliptic. Then $p(x,\xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x,\xi) = \chi(\xi)p(x,\xi)^{-1}$$

$|q(x,\xi)| \le C(1+|\xi|)^{-m}$

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So if

$$Qu(x) = \frac{1}{(2\pi)^n}\int e^{i(x-x')\cdot\xi} q(x,\xi)u(x')\,dx'\,d\xi$$

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$PQ = \mathrm{I} - R$ where R improves Sobolev regularity by 1.

Suppose $q(x, \xi)$ is smooth and

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)\right| \leq C(1 + |\xi|)^{m-|\beta|}$$

for some m. Then

$$\int e^{i(x-x')\cdot\xi} q(x, \xi)\, d\xi$$

has singularities only on $x = x'$:

$$(x - x')^\beta e^{i(x-x')\cdot\xi} = D_\xi^\beta e^{i(x-x')\cdot\xi}$$

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$$Qf(p) = \int_{\mathcal{M}} K(p, p')\, d\mathfrak{m}(p')$$

In $\mathring{\mathcal{M}} \times \mathring{\mathcal{M}}$ we know that K_Q is locally given by

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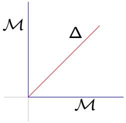

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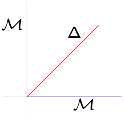

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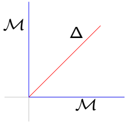

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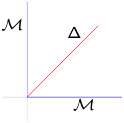

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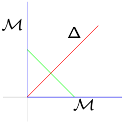

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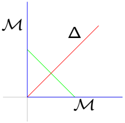

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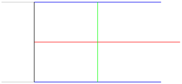

This results in a new smooth manifold with corners, $\mathcal{M} \check{\times} \mathcal{M}$.

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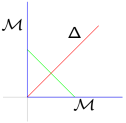

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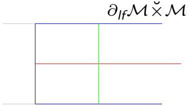

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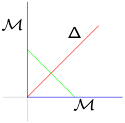

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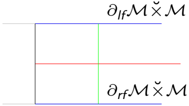

This results in a new smooth manifold with corners, $\mathcal{M}\check{\times}\mathcal{M}$. It has a left face, a right face,

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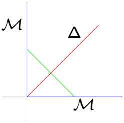

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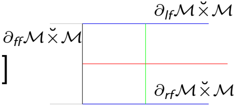

This results in a new smooth manifold with corners, $\mathcal{M}\check{\times}\mathcal{M}$. It has a left face, a right face, and a "new" face, the "front face" $r = 0$.

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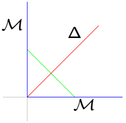

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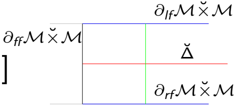

This results in a new smooth manifold with corners, $\mathcal{M}\check{\times}\mathcal{M}$. It has a left face, a right face, and a "new" face, the "front face" $r = 0$. The lifting of the diagonal in $\mathring{\mathcal{M}} \times \mathring{\mathcal{M}}$ has closure

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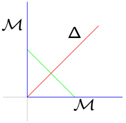

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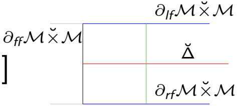

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It intersects $\partial_{ff}\mathcal{M}\check{\times}\mathcal{M}$ transversally.

Let $\wp : \mathcal{M} \check{\times} \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the blow-down map. View the operator P as acting in the first factor in $\mathcal{M} \times \mathcal{M}$.

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$$\mathcal{M} \times \mathcal{M} \xrightarrow{\pi_L} \mathcal{M}$$

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$$\begin{array}{ccc} & & \mathcal{M} \\ & \nearrow^{\pi_L} & \\ \mathcal{M} \times \mathcal{M} & & \\ & \searrow_{\pi_R} & \\ & & \mathcal{M} \end{array}$$

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$$\mathcal{M}\check{\times}\mathcal{M} \xrightarrow{\wp} \mathcal{M} \times \mathcal{M} \begin{array}{l} \xrightarrow{\pi_L} \mathcal{M} \\ \xrightarrow[\pi_R]{} \mathcal{M} \end{array}$$

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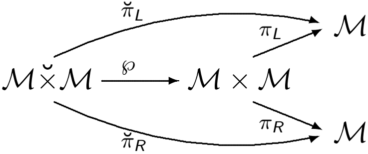

Lift P through the left factor. Since

$$x = \frac{r(1+s)}{2}, \quad x' = \frac{r(1-s)}{2},$$

we have

$$x\partial_x = \wp_*\big(\tfrac{1}{2}[(1+s)r\partial_r + (1-s^2)\partial_s]\big),$$
$$x'\partial_{x'} = \wp_*\big(\tfrac{1}{2}[(1-s)r\partial_r - (1-s^2)\partial_s]\big)$$

and $\wp_*(\partial y^j) = \partial y^j$, etc.

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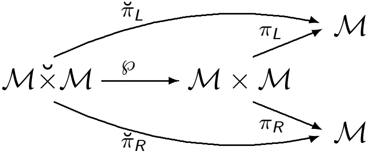

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$$\breve{\pi}_L^* P = \sum\nolimits_{k+|\alpha|\le m} a_{k,\alpha}((1+s)r, y)\big(\tfrac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\big)^k D_y^\alpha$$

From previous slide,

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The principal symbol of $\breve{\pi}_L^* P$ is

$$\sum_{k+|\alpha|=m} a_{k\alpha}((1+s)r, y)\Big(\frac{1}{2}[(1+s)\rho + (1-s^2)\sigma]\Big)^k \eta^\alpha$$

Setting $s=0$ and $\rho=0$ (to see what happens on the conormal bundle of the lifted diagonal) we get:

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extends smoothly across $s = 0$. So there is a distribution $\breve{K}_Q$ on $\mathcal{M}\breve{\times}\mathcal{M}$ which is the restriction of a distribution on an extension of $\mathcal{M}\breve{\times}\mathcal{M}$ across $\partial_{ff}\mathcal{M}\breve{\times}\mathcal{M}$ supported near $\breve{\Delta}$ such that

$$\breve{\pi}_L^* P \breve{K}_Q = \delta(s)\delta(y-y') - \breve{K}_R$$

where $\breve{K}_R$ is smooth. The distribution $\breve{K}_Q$ and function $\breve{K}_R$ descend form the interior of $\mathcal{M}\breve{\times}\mathcal{M}$ to $\mathring{\mathcal{M}} \times \mathring{\mathcal{M}}$, give a distribution and function K_Q, K_R such that

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The operator $x^\nu Q x^{-\nu}$ maps $x^\nu H_b^t$ to $x^\nu H_b^{t+m}$ for arbitrary t and $x^\nu R x^{-\nu}$ maps $x^\nu H_b^t$ to $x^\nu H_b^\infty$.

Asymptotics

Suppose P is b-elliptic and $u \in x^\nu H_b^s$ is such that $Pu = 0$ (or $Pu \in x^\infty H_b^\infty$). Then $u \in x^\nu H_b^\infty$: With Q and R as just constructed,

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The operator $x^{-\mu} Q x^\mu$ is of the same kind as Q, so

$$x^{-\nu} Qf = x^{-\nu} Q x^\nu (x^{-\nu} f) \in H_b^\infty \quad \text{since } x^{-\nu} f \in H_b^\infty$$

Since $Ru \in x^\nu H_b^\infty$,

$$u = Qf + Ru \in x^\nu H_b^\infty.$$

Suppose $u \in x^\nu L^2_b$ and $Pu \in x^\infty H^\infty_b$. Then (assume u supported near $\partial\mathcal{M}$)

$$\mathscr{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}$$

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$$u(x,y) = \frac{1}{2\pi} \int_{\mathrm{Im}\,\sigma = t} x^{i\sigma} \mathscr{M}(u)(\sigma, y) d\sigma$$

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with $t \geq -\nu$. We have $P(u) \in \dot{C}^\infty(\mathcal{M})$ so $\mathscr{M}(Pu)$ is entire. Using the Taylor expansion of P at $x = 0$:

$$P = \sum_{\ell=0}^{N} x^\ell P_\ell(y, xD_x, D_y) + x^{N+1} \tilde{P}_{N+1}(x, y, xD_x, D_y)$$

we get

$$\mathscr{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma + i\ell)} P_\ell(y, xD_x, D_y) u(x,y) \frac{dx}{x} + \int x^{-i(\sigma + i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x,y) \frac{dx}{x}$$

The left hand side of

$$\mathscr{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x,y) \, \frac{dx}{x} + \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x,y) \, \frac{dx}{x}$$

is entire. Since $P_\ell(y, xD_x, D_y)u \in x^\nu H_b^\infty$, the term

$$\int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x,y) \, \frac{dx}{x} = P_\ell(y, \sigma, D_y) \int x^{-i(\sigma+i\ell)} u(x,y) \, \frac{dx}{x}$$

is holomorphic in $\operatorname{Im}(\sigma + i\ell) > -\nu$, that is, $\operatorname{Im}\sigma > -\nu - \ell$. Likewise the reminder gives a term holomorphic in $\operatorname{Im}\sigma > -\nu - N - 1$.

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is holomorphic in $\mathrm{Im}\,(\sigma + i\ell) > -\nu$, that is, $\mathrm{Im}\,\sigma > -\nu - \ell$. Likewise the reminder gives a term holomorphic in $\mathrm{Im}\,\sigma > -\nu - N - 1$. So (with $N = 0$):

$$P_0(y, \sigma, D_y)\mathscr{M}(u)(\sigma) = \mathscr{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y)u(x,y)\frac{dx}{x}$$

is holomorphic in $\mathrm{Im}\,\sigma > -\nu - 1$. But $P_0(y, \sigma, D_y) = \widehat{P}(\sigma)$.

From previous slide:

$$P_0(y,\sigma,D_y)\mathscr{M}(u)(\sigma)=\mathscr{M}(Pu)(\sigma)-\int x^{-i(\sigma+i)}\tilde{P}_1(x,y,xD_x,D_y)u(x,y)\frac{dx}{x}$$

is holomorphic in $\operatorname{Im}\sigma > -\nu - 1$ and $P_0(y,\sigma,D_y) = \widehat{P}(\sigma)$. So

$$\mathscr{M}(u)(\sigma)=\widehat{P}(\sigma)^{-1}\Big[\mathscr{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)}\tilde{P}_1(x,y,xD_x,D_y)u(x,y)\frac{dx}{x}\Big]$$

which is holomorpic in $\operatorname{Im}\sigma > -\nu$, extends as a meromorphic function to $\operatorname{Im}\sigma > -\nu - 1$ with poles in $\operatorname{spec}_b(P) \cap \{-\nu - 1 < \operatorname{Im}\sigma < \nu\}$.

In general,

$$\mathscr{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} P_\ell(y,\sigma,D_y)\mathscr{M}(u)(\sigma+i\ell) + \int x^{-i(\sigma+i(N+1))}\tilde{P}_{N+1}(x,y,xD_x,D_y)u(x,y)\,\frac{dx}{x}$$

gives

$$\widehat{P}(\sigma)\mathscr{M}(u)(\sigma) = \mathscr{M}(Pu)(\sigma) - \sum_{\ell=1}^{N} P_\ell(y,\sigma,D_y)\mathscr{M}(u)(\sigma+i\ell) - \int x^{-i(\sigma+i(N+1))}\tilde{P}_{N+1}(x,y,xD_x,D_y)u(x,y)\,\frac{dx}{x}$$

Once one has shown that $\mathscr{M}(u)(\sigma)$ has a meromorphic extension to $\operatorname{Im}\sigma > -\nu - N$, the right hand side is meromorphic in $\operatorname{Im}\sigma > -\nu - N - 1$, and so $\mathscr{M}(u)(\sigma)$ has a meromorphic extension to $\operatorname{Im}\sigma > -\nu - N - 1$.

The poles of $\mathscr{M}(u)$ are contained in

$$\{\sigma - i\ell : \sigma \in \mathrm{spec}_b(P),\ \operatorname{Im}\sigma < -\nu,\ k = 0, 1, \dots\}$$

Using the Mellin inversion formula:

$$u(x,y) = \frac{1}{2\pi}\int_{Im\sigma = t} x^{i\sigma}\mathscr{M}(u)(\sigma, y)\, d\sigma$$

(with some $t > -\nu$) one gets

$$u(x,y) = \sum_{\substack{\sigma \in \mathrm{spec}_b(P) \\ t-s < Im\sigma - \ell < -\nu}} \sum_{k=0}^{N_{\sigma,\ell}} x^{i\sigma+\ell} u_{\sigma,\ell,k}(y) \log^k x + \frac{1}{2\pi}\int_{Im\sigma = t-s} x^{i\sigma}\mathscr{M}(u)(\sigma,y)\, d\sigma$$

($t - s \notin \mathrm{spec}_b(P)$ and ℓ runs through nonnegative integers). The reminder is $\mathcal{O}(x^{-\nu-s+\varepsilon})$ and the $u_{\sigma,\ell,k}(y)$ are smooth.

Thus:

Suppose $P \in \mathrm{Diff}_b^m(\mathcal{M}; E, F)$ is b-elliptic and $u \in x^\nu H_b^s(\mathcal{M}; E)$ is such that $Pu \in \dot{C}^\infty(\mathcal{M}; F)$. Then $u \in x^\nu H_b^\infty(\mathcal{M}; E)$ and

$$u(x,y) \sim \sum_{\substack{\sigma \in \mathrm{spec}_b(P) \\ Im\sigma < -\nu}} \sum_{\ell=0}^{\infty} \sum_{k=0}^{N_{\sigma,\ell}} x^{i\sigma+\ell} u_{\sigma,\ell,k}(y) \log^k x$$

with $u_{\sigma,\ell,k} \in C^\infty(\mathcal{N}; E)$.

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