Analysis on simply stratified complex manifolds

II. Differential topology of stratified manifolds

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Partial Differential Equations in Complex Geometry

and Singular Spaces,

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A number of problems in partial differential equations on stratified or noncompact manifolds become amenable to treatment after a process of (real) “resolution” of singularities or of compactification. The lectures will describe the differential topology of the resulting spaces, the various kinds of differential operators that arise, and give an overview of some results of the theory.

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In this second lecture I will describe a variety of structures that arise for which related differential operators can be studied with ideas more or less uniform across the various situations.

A fuller dicussion of stratified manifolds was ommitted due to time constraints.

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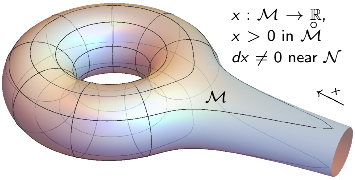
In general, if $\mathcal{M}$ is a manifold with boundary $\mathcal{N}$ then $V \in \mathcal{V}_b$ if and only if in any coordinate system $x, y_1, \dots, y_{n-1}$ near the boundary ($x > 0$ in $\mathring{\mathcal{M}}$),

$$V = a_0 x \frac{\partial}{\partial x} + \sum_{j=1}^{m} a_j \frac{\partial}{\partial y_j}$$



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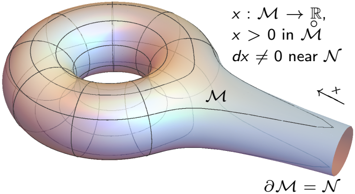

These vector fields can be realized as the image by a suitable map, of the space of sections of another vector bundle. This is essentially based on a theorem of Serre in the analytic category, adapted by Swan to the continuous category.

Specifically...

Theorem. Let $\mathcal{M}$ be a connected paracompact smooth manifold, let $\mathscr{E}$ be a module over $C^\infty(\mathcal{M})$. Suppose $\mathscr{E}$ is locally free finitely generated. Then there is a vector bundle $E \to \mathcal{M}$ such that $\mathscr{E}$ is isomorphic to $C^\infty(\mathcal{M}; E)$.

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$\mathscr{E}$ locally free finitely generated means:

for every $p \in \mathcal{M}$ there is $r \in \mathbb{N}$, a neighborhood U of p, $\eta_1, \dots, \eta_r \in \mathscr{E}$, and $\chi \in C_c^\infty(U)$ with $\chi(p) \neq 0$ such that for any $\phi \in \mathscr{E}$:

a) there are $f^1, \dots, f^r \in C^\infty(\mathcal{M})$ such that $\chi\phi = \sum_{j=1}^r f^j \eta_j$;

b) if $\sum_{j=1}^r f^j \eta_j = \chi\phi = \sum_{j=1}^r \tilde{f}^j \eta_j$, then $f_j = \tilde{f}_j$ on $\operatorname{supp} \chi$ for each $j = 1, \dots, r$.

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Theorem. Let $F \to \mathcal{M}$ be a smooth vector bundle and $\mathscr{E} \subset C^\infty(\mathcal{M}; F)$ a submodule. If $\mathscr{E}$ is locally free finitely generated and $E \to \mathcal{M}$ is a vector bundle such that $\mathscr{E} \approx C^\infty(\mathcal{M}; E)$, then there is a unique bundle homomorphism $\iota : E \to F$ such that $\iota_* : C^\infty(E; \mathcal{M}) \to C^\infty(\mathcal{M}; F)$ is an isomorphism onto $\mathscr{E}$.

b-manifolds

Let $\mathcal{M}$ be a compact manifold with boundary $\mathcal{N}$, let $\mathcal{V}_b \subset C^\infty(\mathcal{M}; T\mathcal{M})$ consist of all vector fields of $\mathcal{M}$ tangent to $\mathcal{N}$. Then $\mathcal{V}_b$ is a module over $C^\infty(\mathcal{M})$.

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If $p \in \mathring{\mathcal{M}}$ and $w_1, \ldots, w_n$ are coordinates about p, then every $V \in \mathcal{V}_b$ is uniquely of the form $\sum b_\mu \partial_{w_\mu}$ near p with smooth b_j.

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This implies that $\mathcal{V}_b$ is the space of sections of a vector bundle, denoted ${}^bT\mathcal{M}$. There is a map ${}^b\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$. which is an isomorphism over $\mathring{\mathcal{M}}$.

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b-manifolds model manifolds with cylindrical ends.

e-manifolds

Again $\mathcal{M}$ is a compact manifold with boundary $\mathcal{N}$, but in addition there is a fibration

$$\begin{array}{ccc} \mathcal{Z} \subset & \mathcal{N} \\ & \downarrow \\ & \mathcal{Y} \end{array}$$

Let $\mathcal{V}_e$ be the subspace of $C^\infty(\mathcal{M}; T\mathcal{M})$ whose element are, over the boundary, tangent to the fibers of $\wp$. This is a locally free finitely generated module over $C^\infty(\mathcal{M})$. The associated vector bundle is ${}^eT\mathcal{M}$.

Θ-structures

Let $\Omega \subset \mathbb{C}^n$ be an open bounded domain with smooth boundary, let $K(z,\zeta)$ be the Schwartz kernel of the Bergman projection:

$$\Pi : L^2(\Omega) \to \mathcal{H}^2(\Omega),\ \Pi(f) = \int_\Omega K(z,\zeta) f(\zeta)\, d\lambda(\zeta).$$

The Bergman metric is

$$g = \sum_{i,j} \frac{\partial^2 \log K_\Delta(z)}{\partial z_i \partial \overline{z}_j}\, dz_i \otimes d\overline{z}_j, \qquad K_\Delta(z) = K(z,z)$$

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Let ρ be a defining function for $\partial\Omega$, positive in Ω. By a theorem of Fefferman there are smooth functions ϕ, ψ near $\overline{\Omega}$ with $\phi > 0$ such that

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Ignoring ψ,

$$g = \frac{n+1}{\rho^2}\Big(\partial\rho \otimes \overline{\partial}\rho - \sum_{i,j} \rho \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j\Big) + \mathcal{O}(1) \text{ as } \rho \to 0.$$

Strict pseudoconvexity: $-\sum \dfrac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} w_i \overline{w}_j > 0$ if $\sum w_j \dfrac{\partial \rho}{\partial z_i} = 0$ & $w \neq 0$

We look for vector fields V_j s.t $[g(V_i, \overline{V}_j)]$ has the least degeneration.

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Must change C^∞ structure to make $\rho^{1/2}$ smooth.

$$g = \frac{n+1}{\rho^2}\Big(\partial\rho \otimes \overline{\partial}\rho - \sum_{i,j}\rho\frac{\partial^2\rho}{\partial z_i\partial\overline{z}_j}dz_i \otimes d\overline{z}_j\Big) + \mathcal{O}(1) \text{ as } \rho \to 0.$$

Strict pseudoconvexity: $-\sum\frac{\partial^2\rho}{\partial z_i\partial\overline{z}_j}w_i\overline{w}_j > 0$ if $\sum w_j\frac{\partial\rho}{\partial z_i} = 0$ & $w \neq 0$

We look for vector fields V_j s.t $[g(V_i, \overline{V}_j)]$ has the least degeneration.

Let $H = \frac{1}{2}\big(\nabla\rho - iJ\nabla\rho\big)$. Then $H = \sum_j \frac{\partial\rho}{\partial\overline{z}_j}\frac{\partial}{\partial z_j}$ Assume $\rho_{z_n} \neq 0$ near $p_0 \in \partial\Omega$. The vector fields

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$$x_j = y_j,\ j = 1, \dots, 2n-1,\ x_{2n} = \rho, \text{ set } \zeta_j = y_j + iy_{j+n},\ j < n.$$

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Now put $\rho = r^2$:

$$\frac{\partial}{\partial\rho} = \frac{1}{2r}\frac{\partial}{\partial r}$$

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$$\langle \Theta, v\rangle = \begin{cases} \langle \theta, v\rangle \text{ if } v \in T\partial\overline{\Omega}_{1/2} \\ 0 \text{ if } v \in \ker d\pi \end{cases}$$

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Define $\mathcal{V}_\Theta \subset C^\infty(\overline{\Omega}_{1/2}; T\overline{\Omega}_{1/2})$ by specifying $V \in \mathcal{V}_\Theta$ if V vanishes on $\partial\overline{\Omega}_{1/2}$ and $\langle \Theta, V\rangle = \mathcal{O}(r^2)$.

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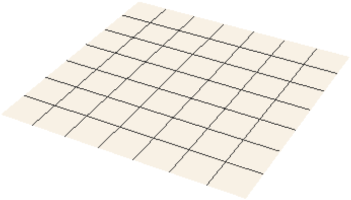

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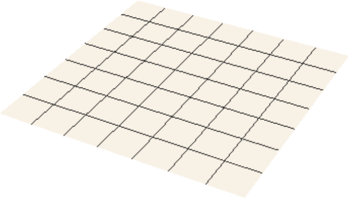

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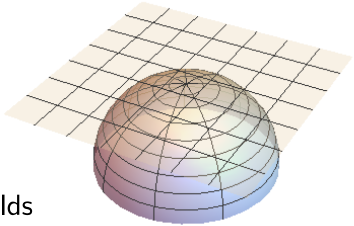

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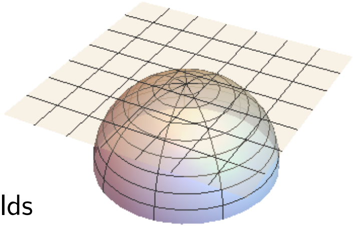

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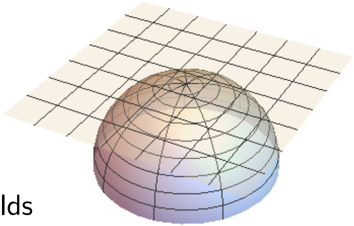

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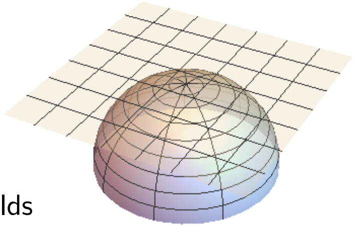

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$\mathcal{V}_{sc}$ locally spanned by $r^2\partial_r$ and $r\partial_{\omega_\ell}$

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We take advantage of this also to define complex structures.

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I'll state classification theorems for such structures generalizing the classification of complex line bundles by their Chern class and of holomorphic line bundles by the Picard group.

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as a direct sum.

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