ON GLOBAL HYPOELLIPTICITY

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Abstract. We consider a first order linear partial differential operator of principal type on a closed connected orientable two-dimensional manifold sending sections of one complex line bundle to sections of another. We prove that the assumption of global hypoellipticity of the operator implies a relation between the degrees of the line bundles and the Euler characteristic of the manifold.

1. Introduction

The main purpose of this note is to establish relations between certain global analytic properties of first order differential operators on two-manifolds and the topology of the underlying objects.

Specifically, we prove a generalization of a theorem of Hounie [5] asserting, in the C^∞ category, that if $\mathcal{M}$ is a closed orientable connected two-manifold and L is a globally defined nowhere zero vector field with complex coefficients that is globally hypoelliptic, then $\mathcal{M}$ is a torus. Of course if $\mathcal{M}$ admits a real nowhere vanishing real vector field, then $\chi(\mathcal{M})$, the Euler characteristic of $\mathcal{M}$ vanishes, so the manifold is a torus; the fact that L has complex coefficients gives Hounie's theorem a content beyond the elementary theory of surfaces. We will also discuss briefly results relevant in this context obtained by the second author and H. Jacobowitz in connection with ellipticity.

Theorem 1.1. *Let $\mathcal{M}$ be an orientable closed connected two-manifold, let $E, F \to \mathcal{M}$ be line bundles, and let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ be a first order (linear) differential operator of principal type. If P is globally hypoelliptic, then*

$$\deg(E) - \deg(F) = \pm\chi(\mathcal{M}). \tag{1.2}$$

The theorem is false if P is a pseudodifferential operator, or if P is not globally hypoelliptic. To be clear, that P is globally hypoelliptic means that if $u \in C^{-\infty}(\mathcal{M}; E)$ and $Pu \in C^\infty(\mathcal{M}; F)$, then in fact $u \in C^\infty(\mathcal{M}; E)$. This is the weakest notion of hypoellipticity in the large. Other (global) notions of hypoellipticity in order of increasing strength are: hypoellipticity at every point, meaning that for any $x \in \mathcal{M}$, if $u \in C^{-\infty}(\mathcal{M}; E)$ and Pu is smooth near x, then u is smooth near x; microlocal hypoellipticity everywhere on $T^*\mathcal{M}\backslash 0$; and finally, ellipticity.

Hounie's theorem corresponds to the case where E is the trivial line bundle (L acts on functions u), as is F (since Lu is also a function), so $\deg E = \deg F = 0$,

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hence $\chi(\mathcal{M}) = 0$. More generally, the same conclusion, that $\mathcal{M}$ is a torus, is reached if E and F are isomorphic (have the same degree). In a slightly different direction, suppose $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is a line subbundle, and let $\mathbb{D} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{V}^*)$ be the induced operator: $\mathbb{D}u$ is the restriction of du to $\mathcal{V}$. If $\mathbb{D}$ is globally hypoelliptic, then $\deg(\mathcal{V}) = \pm\chi(\mathcal{M})$. Again Hounie's theorem follows, since in his case the hypothesis is that $\mathcal{V}$ is trivial.

In the theorem the term *principal type* refers to its original meaning (see Hörmander [2]) adapted as follows. We say that a pseudodifferential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ is of principal type if all of its local expressions in terms of trivializations are of principal type (E and F are line bundles). That is, if for every open set $U \subset \mathcal{M}$ over which E and F admit local frames ϕ and ψ, the scalar operator P_0 such that $P(h\phi) = P_0(h)\psi$ for every $h \in C_c^\infty(U)$, is of principal type in the sense that $d_\xi\boldsymbol{\sigma}(P_0)$, the differential of the principal symbol of P_0 in the fiber direction, does not vanish on $\mathrm{Char}(P)$. If P is a first order differential operator, its principal symbol can be viewed as a linear map $\boldsymbol{\sigma}(P)_x : \mathbb{C}T_x^*\mathcal{M} \to \mathrm{Hom}(E_x, F_x)$ for every $x \in \mathcal{M}$, and P is of principal type if and only if for no x is $\boldsymbol{\sigma}(P)_x$ the zero map. In other words, with the interpretation that $\boldsymbol{\sigma}(P)$ is a section of $\mathrm{Hom}(\mathbb{C}T^*\mathcal{M}, \mathrm{Hom}(E, F))$, principal type means that $\boldsymbol{\sigma}(P)$ is nowhere 0. We use this meaning of *principal type* throughout this note.

Complementing the theorem we also show:

Proposition 1.3. *If $\mathcal{M}$, E and F are as in the hypotheses of the theorem and* (1.2) *holds, then there is a first order globally hypoelliptic differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type. In fact P may be chosen to be elliptic.*

2. First order operators of principal type

Let $\mathcal{M}$ be a smooth manifold of arbitrary dimension and $E, F \to \mathcal{M}$ complex line bundles.

Lemma 2.1. *Suppose $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ is a linear first order differential operator of principal type (in the sense of the introduction). Then P determines a line subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ such that*

$$F \otimes \mathcal{V} \otimes E^* \text{ is trivial.} \tag{2.2}$$

Proof. If ϕ and η are frames of E and F respectively, over some open set $U \subset \mathcal{M}$, then

$$P(u\phi) = (Lu + au)\eta$$

on U for some vector field L and function a. By hypothesis, the principal symbol of L,

$$\xi \mapsto i\langle \xi, L\rangle,$$

does not vanish identically on any fiber of $\mathbb{C}T\mathcal{M}$ over U. Thus L is nowhere 0 on U. If $\tilde{\phi}$ and $\tilde{\eta}$ is another choice of frames and $\tilde{\phi} = \alpha\phi$ and $\tilde{\eta} = \beta\eta$, then

$$P(\tilde{u}\tilde{\phi}) = (\tilde{L}\tilde{u} + \tilde{a}\tilde{u})\tilde{\eta}.$$

Using Leibnitz's formula,

$$P(\tilde{u}\tilde{\phi}) = P(\tilde{u}\alpha\phi) = \big(\alpha L\tilde{u} + (a\alpha + L\alpha)\tilde{u}\big)\eta,$$

therefore

$$\tilde{L} = \frac{\alpha}{\beta}L.$$

Hence P determines a complex line subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ as claimed.

Let ϕ^* and $\tilde{\phi}^*$ be the frames dual to ϕ and $\tilde{\phi}$, respectively. Then $\eta \otimes L \otimes \phi^* = \tilde{\eta} \otimes \tilde{L} \otimes \tilde{\phi}^*$. Indeed,

$$\tilde{\eta} \otimes \tilde{L} \otimes \tilde{\phi}^* = \beta\eta \otimes \frac{\alpha}{\beta} L \otimes \frac{1}{\alpha}\phi^* = \eta \otimes L \otimes \phi^*.$$

Thus $F \otimes \mathcal{V} \otimes E^*$ is trivial, again as asserted. □

It follows in particular that the first Chern classes of E, F, and $\mathcal{V}$ satisfy

$$\mathrm{c}_1(F) + \mathrm{c}_1(\mathcal{V}) - \mathrm{c}_1(E) = 0. \tag{2.3}$$

Conversely:

Lemma 2.4. *Suppose the line bundles $E, F \to \mathcal{M}$ are given. Then:*

(i) there is a line subbundle

$$\iota : \mathcal{V} \hookrightarrow \mathbb{C}T\mathcal{M} \tag{2.5}$$

such that (2.2) *holds, and*

(ii) there is a first order differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type whose associated line bundle is $\mathcal{V}$.

Proof. Regardless of the dimension of $\mathcal{M}$, the line bundle $F^* \otimes E \to \mathcal{M}$ can be realized as a subbundle of $\mathbb{C}T\mathcal{M}$, see [6, Theorem 7.1] or [8, Theorem 2.2]. So let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be isomorphic to $F^* \otimes E$. Then, because these are line bundles, $(F \otimes E^*) \otimes \mathcal{V}$ is trivial, that is, $F \otimes \mathcal{V} \otimes E^*$ is trivial. This proves (i).

To prove (ii) let $\Phi : E \to F \otimes \mathcal{V}$ be a bundle isomorphism. Using the dual

$$\iota^* : \mathbb{C}T^*\mathcal{M} \to \mathcal{V}^* \tag{2.6}$$

of (2.5) we get a map $F \otimes \mathcal{V} \otimes \mathbb{C}T^*\mathcal{M} \to F \otimes \mathcal{V} \otimes \mathcal{V}^*$. Since $\mathcal{V} \otimes \mathcal{V}^*$ is (canonically) trivial, in all we have a bundle homomorphism

$$\Psi : E \otimes \mathbb{C}T^*\mathcal{M} \to F.$$

Pick arbitrarily a connection $\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^*\mathcal{M})$. Using the homomorphism Ψ, if φ is a smooth section of E, then $\Psi\nabla\varphi$ is a section of F. This defines a first order differential operator $Q = \Psi \circ \nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type. We'll leave it to the reader to verify that the subbundle of $\mathbb{C}T\mathcal{M}$ associated with Q by Lemma 2.1 is $\mathcal{V}$. □

Concerning uniqueness we have:

Proposition 2.7. *Fix the line bundles $E, F \to \mathcal{M}$. The set of homotopy classes of first order differential operators $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type is in one-to-one correspondence with $H^1(\mathcal{M}, \mathbb{Z})$.*

Proof. Let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be the line subbundle associated to a first order differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type. So $\mathcal{V}$ is isomorphic to $F^* \otimes E$. If $\mathcal{V}' \subset \mathbb{C}T\mathcal{M}$ is another subbundle isomorphic to $F^* \otimes E$, then again by [6, Theorem 7.1] or [8, Theorem 2.2], $\mathcal{V}'$ is homotopic to $\mathcal{V}$. So the line bundle associated to another first order differential operator $P' : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ of principal type is homotopic to that of P. Suppose now that $\mathcal{V} = \mathcal{V}'$. Then the sections p, p' of $F \otimes \mathcal{V} \otimes E^*$ determined by P and P' as in the proof of Lemma 2.1 are related through a smooth nonvanishing function f giving $fp' = p$. Hence $P - fP'$ is of order 0 and it follows that $P = fP' + a$ for some homomorphism

$a : E \to F$. The proposition now follows from observing that the (multiplicative) group of homotopy classes of continuous (or smooth) functions $f : \mathcal{M} \to \mathbb{C}\backslash 0$ is isomorphic to $H^1(\mathcal{M}, \mathbb{Z})$ and every homomorphism $a : E \to F$ is homotopic to the zero homomorphism. □

A typical example of a first order differential operator of the kind being discussed, already mentioned in the introduction, is as follows. Let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be a line subbundle, and let $\mathbb{D} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{V}^*)$ be the operator $\iota^* \circ d$, where ι^* is as in (2.6). In other words, if f is a smooth function, then $\mathbb{D}f$ is the section of $\mathcal{V}^*$ such that $\langle \mathbb{D}f, v\rangle = vf$ for every $v \in \mathcal{V}$. In this context, Theorem 1.1 gives:

Corollary 2.8. *Let $\mathcal{M}$ be a closed orientable connected two-manifold and $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be a complex line subbundle. If the associated operator $\mathbb{D} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{V}^*)$ is globally hypoelliptic then the degree of $\mathcal{V}$ is $\pm\chi(\mathcal{M})$.*

For example, one can take $\mathcal{V} = T^{0,1}\mathcal{M}$ for some complex structure on $\mathcal{M}$, then $\mathbb{D} = \overline{\partial}$ and $\deg(\mathcal{V}) = -\chi(\mathcal{M})$; this example is an elliptic operator, see Section 6.

Lemma 2.9. *Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ be a linear first order differential operator of principal type. Then the vector subbundle of $\mathbb{C}T\mathcal{M}$ determined by the transpose of P,*

$$ {}^{\dagger}P : C^\infty(\mathcal{M}; F^* \otimes \Omega) \to C^\infty(\mathcal{M}; E^* \otimes \Omega) \tag{2.10}$$

is the same as that determined by P.

By Ω here and below we mean the density bundle of $\mathcal{M}$. The proof of the lemma consists of writing P using frames as in the proof of Lemma 2.1, integrating by parts, and observing that the principal part of ${}^{\dagger}P$ is essentially the negative of the principal part of P.

3. Hypoellipticity and solvability

Let $E, F \to \mathcal{M}$ be complex vector bundles over the orientable closed manifold $\mathcal{M}$ and $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ a differential operator of order m.

Lemma 3.1. *Suppose P is globally hypoelliptic. Then for every $s, r \in \mathbb{R}$ there are $t \in \mathbb{R}$ and $C > 0$ such that*

$$\|v\|_r \le C(\|Pv\|_t + \|v\|_s) \quad \forall v \in C^\infty(\mathcal{M}; E). \tag{3.2}$$

Proof taken from [10]*, p. 538.* Fix $s \in \mathbb{R}$ arbitrarily. Define

$$\wp_t(v) = \|Pv\|_t + \|v\|_s, \quad v \in C^\infty(\mathcal{M}; E).$$

This family of norms defines a metrizable topology on $C^\infty(\mathcal{M}; E)$. Let Φ^s_0 be the space $C^\infty(\mathcal{M}; E)$ with the topology defined by the seminorms $\wp_t$, and let Φ^s be its completion. The inclusion map $\Phi^s_0 \hookrightarrow H^s$ is continuous, hence it extends to a continuous map $\iota : \Phi^s \to H^s$ which is again injective because of the hypoellipticity of P. Thus Φ^s can be viewed as a subspace of H^s. If $v \in \Phi^s$, then $\|Pv\|_t$ is finite for every t, hence $Pv \in C^\infty(\mathcal{M}; F)$ and thus $v \in C^\infty(\mathcal{M}; E)$. Therefore Φ^s is equal to $C^\infty(\mathcal{M}; E)$ as a vector space. The identity map $C^\infty(\mathcal{M}; E) \to \Phi^s$ is continuous and by what was just proved, surjective, hence open. Consequently, $C^\infty(\mathcal{M}; E)$ and Φ^s are isomorphic topological vector spaces. It follows that for every r there are C and t such that (3.2) holds. □

Choosing $r \le s$ in the lemma leads to no information. Suppose $s < r$. Using $\|Pv\|_t \le c\|v\|_{m+t}$ in (3.2) gives $\|v\|_r \le C(\|v\|_{m+t} + \|v\|_s)$, which implies $\|v\|_r \le C\|v\|_{\max\{m+t,s\}}$ for every $v \in C^\infty$. It follows that $r \le \max\{m+t, s\}$. We deduce $s < m+t$ and $r \le m+t$ since $r > s$.

The most convenient definition of solvability is the following:

Definition 3.3 ([3], p. 70)**.** Let $\mathcal{U}$ be a manifold (compactness of $\mathcal{U}$ is not assumed), E, $F \to \mathcal{U}$ vector bundles, $P : C^\infty(\mathcal{U}; E) \to C^\infty(\mathcal{U}; F)$ a differential operator, and $K \subset \mathcal{U}$ a compact subset. Then P is solvable at K if there are $f_1, \dots, f_N \in C^\infty(\mathcal{U}; F)$ such that for every $f \in C^\infty(\mathcal{U}; F)$ there are $u \in C^{-\infty}(\mathcal{U}; E)$ and $c_1 \dots, c_N \in \mathbb{C}$ such that

$$Pu = f + \sum c_j f_j$$

near K.

An immediate consequence of the definition is that if $\mathcal{M}$ is closed (as it always is in this section) and P is a differential operator that is solvable on $\mathcal{M}$, then P is solvable at every compact subset of every open subset $\mathcal{U} \subset \mathcal{M}$.

Lemma 3.4. *If P is globally hypoelliptic, then ${}^\dagger P : C^\infty(\mathcal{M}; F^* \otimes \Omega) \to C^\infty(\mathcal{M}; E^* \otimes \Omega)$ is globally solvable.*

The proof is a minor modification of the arguments in [1, p. 206].

Proof. Since P is hypoelliptic, $\ker P \subset C^\infty(\mathcal{M}; E)$. Fix s and $r \in \mathbb{R}$ arbitrarily with $s < r$ and let $t \in \mathbb{R}$ and $C > 0$ be such that (3.2) holds. Then $\|v\|_r \le C\|v\|_s$ when $v \in \ker P$, so the latter space is finite-dimensional. Let $W \subset H^r$ be the orthogonal of $\ker P \subset H^r$. Then there is C such that

$$\|v\|_r \le C\|Pv\|_t \quad \text{if } v \in W \cap C^\infty. \tag{3.5}$$

If not, there is a sequence v_j in C^∞ such that $\|v_j\|_r = 1$ but $Pv_j \to 0$ in H_t. Passing to a subsequence, we can assume that v_j converges weakly in H_r, hence strongly in H_s, to some v. Then Pv_j converges to Pv, so $v \in \ker P$. In particular $v \in C^\infty \subset H^r$. But (3.2) gives $1 \le C\|v\|_s$, which implies that $v \ne 0$ as an element of H^r. Since $(v_j, v)_r = 0$ for all j and $v_j \to v$ weakly, $(v, v)_r = 0$, a contradiction. Thus (3.5) holds.

Suppose now that $f \in C^\infty(\mathcal{M}; E^* \otimes \Omega)$ is such that $\langle f, v\rangle = 0$ whenever $v \in \ker P$. If $v \in W \cap C^\infty$, then

$$|\langle f, v\rangle| \le \|f\|_{-r}\|v\|_r \le C\|f\|_{-r}\|Pv\|_t,$$

hence

$$|\langle f, v\rangle| \le C\|f\|_{-r}\|Pv\|_t \quad \text{for all } v \in C^\infty$$

because adding an element of $\ker P$ to v does not change either side of the inequality. We conclude that the linear form

$$C^\infty(\mathcal{M}; F) \supset PC^\infty(\mathcal{M}; E) \ni Pv \mapsto \langle f, v\rangle \in \mathbb{C}$$

is continuous, hence there is $u \in C^{-\infty}(\mathcal{M}; F^* \otimes \Omega)$ such that

$$\langle f, v\rangle = \langle u, Pv\rangle \quad \text{for all } v \in C^\infty$$

which means that ${}^\dagger P u = f$. $\square$

4. Solvability and Condition ($\mathscr{P}$)

We have seen that if $\mathcal{M}$ is closed and $P : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};F)$ is globally hypoelliptic, then ${}^\dagger P$ is globally solvable. In particular, ${}^\dagger P$ is solvable at every compact subset of every open subset $\mathcal{U} \subset \mathcal{M}$. If P happens to be a scalar operator of principal type, this implies that ${}^\dagger P$, hence P, satisfies Condition ($\mathscr{P}$):

($\mathscr{P}$) For every elliptic symbol q, $\Im(q\boldsymbol{\sigma}(P))$ does not change sign along any complete integral curve of the Hamiltonian vector field of $\Re(q\boldsymbol{\sigma}(P))$ passing through a point in $\mathrm{Char}(P)$.

If $P : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};F)$ is a pseudodifferential operator acting on line bundles, its principal symbol is generally not a scalar function: if $\pi : T^*\mathcal{M}\backslash 0 \to \mathcal{M}$ is the projection, then $\pi^*F \otimes \pi^*E^*$ need not be a trivial line bundle. So ($\mathscr{P}$) does not generally make sense when P is an operator acting on line bundles. The condition does make sense locally, via trivializations, but the validity of Condition ($\mathscr{P}$) locally (for scalar operators) does not imply its global validity.

In the case of a line subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{U}$ on a manifold $\mathcal{U}$ there is an alternative definition of Condition ($\mathscr{P}$) which goes as follows ([11, Definition (VIII.7.1)]):

($\mathscr{P}'$) The orbits of $\mathcal{V}$ are of dimension ≤ 2, the two-dimensional orbits of $\mathcal{V}$ are orientable, and whenever L is a section of $\mathcal{V}$ and $\mathcal{O}$ is a two-dimensional orbit of $\mathcal{V}$, then the restriction of $iL \wedge \overline{L}$ to $\mathcal{O}$, a section of $\bigwedge^2 T\mathcal{O}$, is a section of the closure of one of the two components of $\bigwedge^2 T\mathcal{O}\backslash 0$.

By an orbit of $\mathcal{V}$ we mean an equivalence class of the relation defined by setting $x \sim x'$ iff there is a continuous, piecewise smooth curve starting at x and ending at x' whose segments are integral curves of the real part of sections of $\mathcal{V}$ (see [9]). These orbits are smooth immersed submanifolds and for any x, if $v \in \mathcal{V}_x$, then $\Re v$ is tangent to the orbit through x (see [9]).

When $\mathcal{M}$ is closed we have the following equivalent definition used by Hounie in [4]. Let $\Re C^\infty(\mathcal{M};\mathcal{V})$ be the set of real parts of smooth sections of $\mathcal{V}$. Each element of $\Re C^\infty(\mathcal{M};\mathcal{V})$ is a smooth real vector field, the infinitesimal generator of a one-parameter group of diffeomorphisms of $\mathcal{M}$. Let $G_\mathcal{V}$ be the group generated by the diffeomorphisms determined by all the elements of $\Re C^\infty(\mathcal{M};\mathcal{V})$. An orbit of $\mathcal{V}$ is then an orbit of $G_\mathcal{V}$.

A minimal set of $\mathcal{V}$ is a closed nonempty subset of $\mathcal{M}$ which is $G_\mathcal{V}$-invariant and has no proper closed invariant subsets. Minimal sets are unions of orbits.

Henceforth we assume that $\mathcal{M}$ is a closed orientable connected two-manifold.

Proposition 4.1 (Hounie [4]). *Let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be a smooth line subbundle. Then the minimal sets of $\mathcal{V}$ are either $\mathcal{M}$ or diffeomorphic to S^1.*

Actually in Hounie's result there is also the possibility of minimal sets consisting of a single point, but that case does not arise here because no orbit is a single point.

Proposition 4.2. *Let $E, F \to \mathcal{M}$ be line bundles, let $P : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};F)$ be a first order differential operator of principal type, and let $\mathcal{V}$ be the associated subbundle of $\mathbb{C}T\mathcal{M}$. If P is globally hypoelliptic, then the only minimal set of $\mathcal{V}$ is $\mathcal{M}$ itself.*

Proof. Global hypoellipticity precludes having a solution of $Pu = f \in C^\infty$ such that u is singular along a minimal set which is diffeomorphic to a circle. Indeed, suppose that γ is a minimal set of $\mathcal{V}$ diffeomorphic to S^1. The line bundles E, F, and $\mathcal{V}$ are trivial along γ, so there is a neighborhood $\mathcal{U}$ of γ over which they are also trivial. Let ϕ, η, L be smooth frames for these line bundles over $\mathcal{U}$. With some nonvanishing function f and some function a we therefore have

$$P(u\phi) = (fLu + au)\eta$$

on $\mathcal{U}$. Since P is of principal type, its solvability is independent of lower order terms. Hence we may assume that a vanishes near γ. We may further assume that $\mathcal{U}\backslash\gamma$ has two components. There is a smooth function u_0 on $\mathcal{U}\backslash\gamma$ such that $u_0 = 1$ on one of these components and $u_0 = 0$ on the other. Let $\chi \in C_c^\infty(\mathcal{U})$ be equal to 1 near γ. We may view $\chi u_0\phi$ as a global section of E and as such its singular support is γ. However, $P(\chi u_0\phi)$ is smooth on $\mathcal{M}$, since L is tangent to γ and a vanishes near γ. This contradicts the assumed hypoellipticity of P, and we conclude that no such γ exists. Hence, by Proposition 4.1, $\mathcal{M}$ is the only minimal set of $\mathcal{V}$. □

Suppose $\mathcal{V}$ is a subbundle of $\mathbb{C}T\mathcal{M}$ for which the conclusion of Proposition 4.2 holds. Then either $\mathcal{M}$ is itself an orbit, or every orbit is one-dimensional and dense. To see this, suppose there is is a one-dimensional orbit $\mathcal{O}$. Its closure contains any minimal set containing $\mathcal{O}$, but such a minimal set must be $\mathcal{M}$ itself. So $\mathcal{O}$ is dense in $\mathcal{M}$. If $x \in \mathcal{O}$ and $v \in \mathcal{V}_x$, then v is tangent to $\mathcal{O}$. Thus v is proportional to a real vector. Since $\mathcal{O}$ is dense, this is true at any point of $\mathcal{M}$: $\mathcal{V}$ is the complexification of a real subbundle of $T\mathcal{M}$. It follows that every orbit of $\mathcal{V}$ is one-dimensional. Thus:

Lemma 4.3. *Suppose $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is a line subbundle whose only minimal set is $\mathcal{M}$. Then either every orbit of $\mathcal{V}$ is one-dimensional (and then $\mathcal{M}$ must be a torus), or $\mathcal{M}$ is itself an orbit. Thus, with the hypotheses of Proposition 4.2, $\mathcal{M}$ is the only orbit of $\mathcal{V}$.*

We are now ready for our key result.

Theorem 4.4. *Let $\mathcal{M}$ be a closed orientable connected two-dimensional manifold, let $E, F \to \mathcal{M}$ be line bundles, let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ be a first order differential operator of principal type, and let $\mathcal{V}$ be its associated line bundle. If P is globally hypoelliptic, then there is a component $\mathcal{C}$ of $\bigwedge^2 T\mathcal{M}\backslash 0$ such that $iv \wedge \overline{v}$ belongs to the closure $\mathrm{Cl}(\mathcal{C})$ of $\mathcal{C}$ for all $v \in \mathcal{V}$.*

The orientability of $\mathcal{M}$ ensures that $\bigwedge^2 T\mathcal{M}\backslash 0$ has two connected components.

Proof. Note that $\mathrm{Cl}(\mathcal{C})$ is just $\mathcal{C}$ union the zero section of $\bigwedge^2 T\mathcal{M}$. If $\mathcal{V}$ has a one-dimensional orbit, then as we just saw, every orbit is one-dimensional and $\mathcal{V}$ is the complexification of a subbundle of the real tangent bundle of $\mathcal{M}$. It follows that if $v \in \mathcal{V}$, then $iv \wedge \overline{v} = 0$, so $iv \wedge \overline{v} \in \mathrm{Cl}(\mathcal{C})$, no matter which component of $T\mathcal{M}\backslash 0$ the set $\mathcal{C}$ actually is.

Suppose now that $\mathcal{V}$ has no one-dimensional orbits. Then $\mathcal{M}$ is the only orbit of $\mathcal{V}$. Let $x_0 \in \mathcal{M}$ be a point where $\mathcal{V}_{x_0}$ is not the complexification of a real subspace of $T_{x_0}\mathcal{M}$. Such x_0 exists because no orbit of $\mathcal{V}$ is one-dimensional. Then, if $v \in \mathcal{V}_{x_0}$ and $v \neq 0$, then $iv \wedge \overline{v}$ is not zero and, because $\mathcal{V}_{x_0}$ is (complex) one-dimensional, must belong to one of the components of $T\mathcal{M}\backslash 0$. Label that component $\mathcal{C}$:

$$iv \wedge \overline{v} \in \mathcal{C} \text{ if } v \in \mathcal{V}_{x_0}. \tag{4.5}$$

Now pick $x \in \mathcal{M}$ arbitrarily; we will show that

$$iv \wedge \overline{v} \in \mathrm{Cl}(\mathcal{C}) \text{ if } v \in \mathcal{V}_x. \tag{4.6}$$

Since $\mathcal{M}$ is an (the) orbit of $\mathcal{V}$, there is a piecewise smooth curve starting at x_0 and ending at x whose segments are integral curves of real parts of smooth sections of $\mathcal{V}$. This curve can be assumed to be simple. In a neighborhood $\mathcal{U}$ of such a curve γ the line bundles E, F, and $\mathcal{V}$ are trivial. The orbit of $\mathcal{V}$ *within* $\mathcal{U}$ contains both x_0 and x, since $\gamma \subset \mathcal{U}$. This orbit, call it $\mathcal{O}$, is two dimensional, because it contains a neighborhood of x_0 due to the fact that $iv \wedge \overline{v} \neq 0$ if $v \in \mathcal{V}_{x_0}$. Let ϕ be a frame of E over $\mathcal{U}$, write ϕ^* for the dual frame, likewise let η be a frame of F over $\mathcal{U}$ and η^* its dual frame. Let μ be a global smooth nonvanishing section of the density bundle of $\mathcal{M}$. Then, with some smooth function b on $\mathcal{U}$,

$${}^{\dagger}P(u\eta^* \otimes \mu) = (Lu + bu)\phi^* \otimes \mu$$

over $\mathcal{U}$ with L a vector field which by Lemma 2.9 is a section of $\mathcal{V}$ again, and a frame because ${}^{\dagger}P$ is of principal type. Since P is globally hypoelliptic, ${}^{\dagger}P$ is solvable at any compact subset of $\mathcal{U}$. It follows that ${}^{\dagger}P$ satisfies Condition $(\mathscr{P})$, hence $\mathcal{V}|_{\mathcal{U}}$ satisfies Condition $(\mathscr{P}')$ on $\mathcal{O}$. The validity of (4.6) now follows from (4.5). Since x is arbitrary, the proof of the theorem is complete. □

5. The degree of the associated bundle

We continue to assume that $\mathcal{M}$ is a closed, orientable, connected manifold of dimension 2. The following lemma completes the proof of Theorem 1.1:

Lemma 5.1. *Suppose $\mathcal{V}$ is a line subbundle of $\mathbb{C}T\mathcal{M}$ such that there is a component $\mathcal{C}$ of $\bigwedge^2 T\mathcal{M}\backslash 0$ such that $iv \wedge \overline{v} \in \mathrm{Cl}(\mathcal{C})$ for all $v \in \mathcal{V}$. Then*

$$\mathrm{c}_1(\mathcal{V}) = \pm\, \mathbf{e}(\mathcal{M}).$$

Here $\mathbf{e}(\mathcal{M})$ is the Euler class of $\mathcal{M}$.

As a consequence, if $\mathcal{V}$ satisfies (2.3), then

$$\mathrm{c}_1(F) \pm \mathbf{e}(\mathcal{M}) - \mathrm{c}_1(E) = 0.$$

Integration over $\mathcal{M}$ yields (1.2). This completes the proof of Theorem 1.1 since its hypotheses ensure the validity of the hypothesis of Lemma 5.1 by way of Theorem 4.4, and since (2.3) holds for E, F, and $\mathcal{V}$ when the latter is the associated vector bundle of P.

Proof (See [8, Theorem 7.3]*).* The subbundle $\mathcal{V}$ determines (and is determined by) a smooth section $[\mathcal{V}]$ of the fiberwise projectivization of $\mathbb{C}T\mathcal{M}$,

$$\begin{array}{ccc} S^2 \hookrightarrow & \mathbb{P}\mathbb{C}T\mathcal{M} & \\ & \downarrow \;\; \curvearrowright & [\mathcal{V}] \\ & \mathcal{M} & \end{array}$$

The projectivization of $T\mathcal{M} \subset \mathbb{C}T\mathcal{M}$ is a hypersurface $\mathcal{S}$ in $\mathbb{P}\mathbb{C}T\mathcal{M}$ which, because $\mathcal{M}$ is orientable, separates $\mathbb{P}\mathbb{C}T\mathcal{M}$. The hypothesis of the lemma is equivalent to the statement that the image of $[\mathcal{V}]$ lies in the closure of one of the components, call it Γ, of $\mathbb{C}T\mathcal{M}\backslash\mathcal{S}$. Consequently, there is a family of smooth subbundles $\mathcal{V}_t$ depending continuously (even smoothly) on $t \in [0,1]$ with $[\mathcal{V}_0] = [\mathcal{V}]$ and $[\mathcal{V}_t] \in \Gamma$ if $t > 0$. Depending on the orientation chosen for $\mathcal{M}$, the subbundle $\mathcal{V}_t$ with $t > 0$

is a complex structure, or its conjugate is. Since the first Chern class of a complex structure (as antiholomophic tangent vectors) on $\mathcal{M}$ is $\mathbf{e}(\mathcal{M})$,

$$\mathrm{c}_1(\mathcal{V}) = \pm\,\mathbf{e}(\mathcal{M}).$$

This completes the proof of the lemma. □

6. Some remarks on ellipticity

It was shown in [7] that if $\mathcal{M}$ is a smooth orientable connected manifold and $E, F \to \mathcal{M}$ are smooth complex vector bundles of the same rank, then the existence of an elliptic pseudo-differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F) \tag{6.1}$$

implies that the total Chern classes of the bundles are related by $\mathrm{c}(F) - \mathrm{c}(E) = k\,\mathbf{e}(\mathcal{M})$ for some $k \in \mathbb{Z}$. Suppose that E and F are line bundles. Then, if $\dim(\mathcal{M}) > 2$, the existence of an elliptic pseudodifferential operator (6.1) implies that E and F are isomorphic to each other. This is also the case if $\dim \mathcal{M} = 2$ and $\mathcal{M}$ is not closed. Finally, when $\mathcal{M}$ is closed and two-dimensional, then the condition

$$\mathrm{c}_1(F) - \mathrm{c}_1(E) = k\,\mathbf{e}(\mathcal{M}) \tag{6.2}$$

is also sufficient for the existence of an elliptic *differential* operator, which is then necessarily of order $|k|$ if $\mathbf{e}(\mathcal{M}) \neq 0$. The number k, which is completely determined if $\chi(\mathcal{M}) \neq 0$, can be shown to be, except for sign, the order of the differential operator. The reader is referred to [7] for details.

The canonical examples of elliptic operators on a closed, connected, orientable manifold of dimension two are the following. Give $\mathcal{M}$ an orientation. Line bundles on $\mathcal{M}$ are classified by their degree. Let $E^1 \to \mathcal{M}$ be a line bundle of degree 1 and write E^{-1} for its dual. Further let E^ℓ denote the $|\ell|$-th tensor power of E^1 if $\ell > 0$ or E^{-1} of $\ell < 0$. Finally let E^0 denote the trivial line bundle. Every line bundle over $\mathcal{M}$ is isomorphic to one of the E^ℓ. Fix a connection ∇ on E^1. Then we get induced connections on each of the E^ℓ,

$$\nabla : C^\infty(\mathcal{M}; E^\ell) \to C^\infty(M; E^\ell \otimes \mathbb{C}T^*\mathcal{M})$$

Fix a complex structure on $\mathcal{M}$, let $\pi^{0,1} : \mathbb{C}T^*\mathcal{M} \to \mathbb{C}T^*\mathcal{M}$ be the projection on $\bigwedge^{0,1}\mathcal{M}$ according to the decomposition $\mathbb{C}T^*\mathcal{M} = \bigwedge^{1,0}\mathcal{M} \otimes \bigwedge^{0,1}\mathcal{M}$. Write also $\pi^{0,1}$ for the map $E^\ell \otimes \mathbb{C}T^{\mathcal{M}} \to E^\ell \otimes \bigwedge^{0,1}\mathcal{M}$ it defines. Then

$$\nabla^{0,1} = \pi^{1,0} \circ \nabla : C^\infty(\mathcal{M}; E^\ell) \to C^\infty(M; E^\ell \otimes {\textstyle\bigwedge}^{0,1}\mathcal{M})$$

is a first order elliptic operator. Since $\deg(\bigwedge^{0,1}\mathcal{M}) = \chi(\mathcal{M})$, $\deg(E^\ell \otimes \bigwedge^{0,1}\mathcal{M}) = \ell + \chi(\mathcal{M})$, so $E^\ell \otimes \bigwedge^{0,1}\mathcal{M}$ is isomorphic to $E^{\ell+\chi(\mathcal{M})}$. So with $E = E^\ell$ and $F = E^{\ell+\chi(\mathcal{M})}$ we get an elliptic first order operator for which (6.2) holds with $k = 1$. Using the projection $\pi^{1,0}$ on $\bigwedge^{1,0}\mathcal{M}$ in a similar way as $\pi^{0,1}$ we get an example, $\nabla^{1,0}$, where (6.2) holds with $k = -1$. Proposition 1.3 is thus proved by exhibiting $\nabla^{0,1}$ or $\nabla^{1,0}$. Composition then gives examples of elliptic differential operators of order $|k|$ for which (6.2) holds with arbitrary k.

Continuing with $\mathcal{M}$ closed, connected, orientable, of dimension two, let $E, F \to \mathcal{M}$ be arbitrary line bundles and $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ a first order globally hypoelliptic differential operator. We showed that the associated line bundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ of P is homotopic to either a holomorphic or an antiholomorphic structure on $\mathcal{M}$. Combining this with Proposition 2.7 we get:

Proposition 6.3. *Every first order globally hypoelliptic differential operator on $\mathcal{M}$ is homotopic to either $f\nabla^{0,1}$ or $f\nabla^{1,0}$ where f is a nonvanishing function on $\mathcal{M}$.*

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