TWO EMBEDDING THEOREMS

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To Leon Ehrenpreis, in memoriam

ABSTRACT. We first consider pairs $(\mathcal{N},\mathcal{T})$ where $\mathcal{N}$ is a closed connected smooth manifold and $\mathcal{T}$ a nowhere vanishing smooth real vector field on $\mathcal{N}$ that admits an invariant metric and show that there is an embedding $F:\mathcal{N}\to S^{2N-1}\subset\mathbb{C}^N$ for some N mapping $\mathcal{T}$ to a vector field of the form $\mathcal{T}'=i\sum_{j=1}^N\tau_j\big(z^j\frac{\partial}{\partial z^j}-\overline{z}^j\frac{\partial}{\partial\overline{z}^j}\big)$ for some $\tau_j\neq 0$.

We further consider pairs $(\mathcal{N},\mathcal{T})$ with the additional datum of an involutive subbundle $\overline{\mathcal{V}}\subset\mathbb{C}T\mathcal{N}$ such that $\mathcal{V}+\overline{\mathcal{V}}=\mathbb{C}T\mathcal{N}$ and $\mathcal{V}\cap\overline{\mathcal{V}}=\operatorname{span}_{\mathbb{C}}\mathcal{T}$ for which there is a section β of the dual bundle of $\overline{\mathcal{V}}$ such that $\langle\beta,\mathcal{T}\rangle=-i$ and

$$X\langle\beta,Y\rangle-Y\langle\beta,X\rangle-\langle\beta,[X,Y]\rangle=0\quad\text{whenever } X,Y\in C^\infty(\mathcal{N};\overline{\mathcal{V}}).$$

Then $\overline{\mathcal{K}}=\ker\beta$ is a CR structure and we give necessary and sufficient conditions for the existence of a CR embedding of $\mathcal{N}$ (with a possibly different, but related, CR structure) into S^{2N-1} mapping $\mathcal{T}$ to $\mathcal{T}'$.

The first result is an analogue of the fact that for any line bundle $L\to\mathcal{B}$ over a compact base there is an embedding $f:\mathcal{B}\to\mathbb{CP}^{N-1}$ such that L is isomorphic to the pullback by f of the tautological line bundle $\Gamma\to\mathbb{CP}^{N-1}$. The second is an analogue of the statement in complex differential geometry that a holomorphic line bundle over a compact complex manifold is positive if and only if one of its tensor powers is very ample.

1. Introduction

Let $\mathscr{F}$ be the family of pairs $(\mathcal{N},\mathcal{T})$ where $\mathcal{N}$ is a closed connected smooth manifold and $\mathcal{T}$ is a smooth nowhere vanishing real vector field on $\mathcal{N}$ admitting an invariant metric. An example of such a pair is the sphere $S^{2N-1}\subset\mathbb{C}^N$ with the vector field $\mathcal{T}'$ in formula (1.2) below.

We will show:

Theorem 1.1. *Let $(\mathcal{N},\mathcal{T})\in\mathscr{F}$. Then there is a positive integer N, an embedding $F:\mathcal{N}\to\mathbb{C}^N$ with image contained in the sphere S^{2N-1}, and positive numbers τ_j such that $F_*\mathcal{T}$ is the vector field*

$$\mathcal{T}'=i\sum_{j=1}^N\tau_j\left(z^j\frac{\partial}{\partial z^j}-\overline{z}^j\frac{\partial}{\partial\overline{z}^j}\right). \tag{1.2}$$

Furthermore, no component function of F is flat at any point of $\mathcal{N}$.

An element $(\mathcal{N},\mathcal{T})\in\mathscr{F}$ is like the circle bundle of a complex line bundle over a closed manifold $\mathcal{B}$ (with $\mathcal{T}$ being the infinitesimal generator of the circle action),

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and the theorem is like the basic ingredient in the classification theorem for line bundles. In our general setting, the orbits of $\mathcal{T}$ need not be compact.

Theorem 1.1 was stated without proof in [14] as Theorem 3.11. The fact that no component of F is flat was used there in an argument involving the Malgrange Preparation Theorem. The complete proof is given here in Section 2.

The statement of our second result requires us recalling some terminology and a few facts. Associated with any involutive subbundle $\mathcal{W}$ of $T\mathcal{N}$ or its complexification $\mathbb{C}T\mathcal{N}$ there is a first order differential cochain complex on the exterior powers of its dual,

$$\cdots \to C^\infty(\mathcal{N};\textstyle\bigwedge^q\mathcal{W}^*) \to C^\infty(\mathcal{N};\textstyle\bigwedge^{q+1}\mathcal{W}^*) \to \cdots$$

where the coboundary operator is given by Cartan's formula for the differential. We review this in more detail below. The complex is elliptic if and only if $\mathcal{W}+\overline{\mathcal{W}} = \mathbb{C}T\mathcal{N}$ (or $= T\mathcal{N}$ if $\mathcal{W}\subset T\mathcal{N}$), in which case $\mathcal{W}$ is referred to as an elliptic structure.

Let $\mathscr{F}_{\mathrm{ell}}$ be the set of triples $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})$ such that $(\mathcal{N},\mathcal{T})\in\mathscr{F}$, $\overline{\mathcal{V}}\subset\mathbb{C}T\mathcal{N}$ is an elliptic structure with $\mathcal{V}\cap\overline{\mathcal{V}} = \operatorname{span}_{\mathbb{C}}\mathcal{T}$, and there is a closed section β of $\overline{\mathcal{V}}^*$ such that $\langle\beta,\mathcal{T}\rangle = -i$. Closed means in the sense of the associated complex, that is, $\overline{\mathbb{D}}\beta = 0$ where $\overline{\mathbb{D}}$ refers to the coboundary operator of the induced complex:

$$V\langle\beta,W\rangle - W\langle\beta,V\rangle - \langle\beta,[V,W]\rangle = 0 \quad \text{for all } V,\ W\in C^\infty(\mathcal{N};\overline{\mathcal{V}}). \tag{1.3}$$

If $\beta,\ \beta'\in C^\infty(\mathcal{N};\overline{\mathcal{V}}^*)$ are two sections as described, we say that β and β' are equivalent if $\beta'-\beta = \overline{\mathbb{D}}u$ with a real-valued function u and write $\boldsymbol{\beta}$ for the class of β. Here $\overline{\mathbb{D}}u$ means the restriction of du to $\overline{\mathcal{V}}$. Observe that necessarily $\mathcal{T}u = 0$.

Suppose that $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathscr{F}_{\mathrm{ell}}$ and that β is a section of $\overline{\mathcal{V}}^*$ as just described. Let

$$\overline{\mathcal{K}}_\beta = \{v\in\overline{\mathcal{V}} : \langle\beta,v\rangle = 0\}$$

Then $\overline{\mathcal{K}}_\beta$ is a CR structure of codimension 1. Let θ_β be the real 1-form that satisfies $\langle\theta_\beta,\mathcal{T}\rangle = 1$ and whose restriction to $\overline{\mathcal{K}}_\beta$ vanishes. Define

$$\operatorname{Levi}_{\theta_\beta}(v,w) = -id\theta_\beta(v,\overline{w}), \quad v,\ w\in\mathcal{K}_{\beta,p},\ p\in\mathcal{N}; \tag{1.4}$$

$\mathcal{K}_\beta$ is the conjugate of $\overline{\mathcal{K}}_\beta$.

A map $F:\mathcal{N}\to\mathbb{C}^N$ we be called equivariant if $F_*\mathcal{T} = \mathcal{T}'$ for some $\mathcal{T}'$ of the form (1.2).

Theorem 1.5. *Suppose that $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathscr{F}_{\mathrm{ell}}$ with $\dim\mathcal{N}\geq 5$. Fix a class $\boldsymbol{\beta}$ as described above. The following are equivalent:*

1. *There is $\beta\in\boldsymbol{\beta}$ and an equivariant CR immersion of $\mathcal{N}$ with the CR structure $\overline{\mathcal{K}}_\beta$ into $\mathbb{C}^N$ for some N.*
2. *There is $\beta'\in\boldsymbol{\beta}$ and an equivariant CR immersion of $\mathcal{N}$ with the CR structure $\overline{\mathcal{K}}_{\beta'}$ into $\mathbb{C}^N$ for some N with image in S^{2N-1}.*
3. *There is $\beta'\in\boldsymbol{\beta}$ such that the CR structure $\overline{\mathcal{K}}_{\beta'}$ is definite.*
4. *There is $\beta'\in\boldsymbol{\beta}$ and an equivariant CR embedding of $\mathcal{N}$ with the CR structure $\overline{\mathcal{K}}_{\beta'}$ into $\mathbb{C}^N$ with image in S^{2N-1} for some N.*

The implication (3) $\Longrightarrow$ (4) is like Kodaira's embedding theorem of Kähler manifolds with integral fundamental form into complex projective space. This is explained in some detail the paragraphs following Example 1.7. The proof of the implication relies on Boutet de Monvel's construction in [4] of an embedding under the same condition, strict pseudoconvexity; this is the only reason for the restriction on the dimension of $\mathcal{N}$ in the hypothesis of the theorem.

Concrete models of manifolds $\mathcal{N}$ with the structure described above are the following.

Example 1.6. Let $\mathcal{N} = S^{2n+1} \subset \mathbb{C}^{n+1}$, let

$$\mathcal{T} = i \sum_{j=1}^{n+1} \tau_j \left(z^j \frac{\partial}{\partial z^j} - \overline{z}^j \frac{\partial}{\partial \overline{z}^j} \right).$$

Then $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$, since $\mathcal{T}$ preserves the standard metric of S^{2n+1}. Suppose all τ_j have the same sign. Let $\overline{\mathcal{K}}$ be the standard CR structure of S^{2n+1} (as a subbundle of $T^{0,1}\mathbb{C}^{n+1}$ along S^{2n+1}). Then $\mathcal{T}$ is transverse to $\overline{\mathcal{K}}$ and $\overline{\mathcal{V}} = \mathcal{K} \oplus \operatorname{span}_{\mathbb{C}} \mathcal{T}$ is involutive. Let θ be the unique real 1-form on S^{2n+1} which vanishes on $\overline{\mathcal{K}}$ and satisfies $\langle \theta, \mathcal{T} \rangle = 1$ and let $\beta = -i\jmath^*\theta$ where $\jmath : \overline{\mathcal{K}} \to \mathbb{C}TS^{2n+1}$ is the inclusion map. Then (1.3) holds. Indeed, if V and W are CR vector fields, then so is $[V, W]$ since $\overline{\mathcal{K}}$ is involutive, and if V is CR then $[V, \mathcal{T}]$ is also CR, so (1.3) holds if V is CR and $W = \mathcal{T}$. Since $\langle \beta, \mathcal{T} \rangle = -i$, $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$. The set of closures of the orbits of $\mathcal{T}$ is a Hausdorff space, an analogue of complex projective space.

Example 1.7. Let $\mathcal{B}$ be a complex manifold, let $E \to \mathcal{B}$ a Hermitian holomorphic line bundle, and let $\rho : \mathcal{N} \to \mathcal{B}$ be its circle bundle. Define

$$\overline{\mathcal{V}} = \{ v \in \mathbb{C}T\mathcal{N} : \rho_* v \in T^{0,1}\mathcal{B} \}. \tag{1.8}$$

This vector bundle satisfies (1) and (2); the vector field $\mathcal{T}$ is the infinitesimal generator of the standard circle action on $\mathcal{N}$. Identifying $\mathcal{N}$ with the bundle of oriented orthonormal bases of the real bundle underlying E, let θ be the connection form of the Hermitian holomorphic connection, a real smooth 1-form with $\langle \theta, \mathcal{T} \rangle = 1$ and $\mathcal{L}_{\mathcal{T}}\theta = 0$, where $\mathcal{L}_{\mathcal{T}}$ is Lie derivative. Let $\iota : \overline{\mathcal{V}} \to \mathbb{C}T\mathcal{N}$ be the inclusion map. Using the dual map $\iota^* : \mathbb{C}T^*\mathcal{N} \to \overline{\mathcal{V}}^*$, let $\beta = -i\iota^*\theta$. Then $\Im\langle \beta, \mathcal{T} \rangle = -1$ and $\overline{\mathbb{D}}\beta = 0$; that β is $\overline{\mathbb{D}}$-closed is equivalent to the statement that θ corresponds to a holomorphic connection. Adding $\overline{\mathbb{D}}u$ to β with u real-valued and $\mathcal{T}u = 0$ corresponds to a change of the Hermitian metric.

In the context of Example 1.7, let $\overline{\mathcal{K}} = \ker \beta$; this is a CR structure. The statement that Levi_θ (as defined in (1.4)) is positive definite is equivalent to the statement that the line bundle $E \to \mathcal{B}$ is negative (Grauert [6], see also Kobayashi [8, p. 87]), that is, the form ω on $\mathcal{B}$ such that $\rho^*\omega = -id\theta$ is the fundamental form of a Kähler metric on $\mathcal{B}$.

Kodaira's embedding theorem [9] asserts that if $\mathcal{B}$ is compact and admits a Kähler metric whose fundamental form is in the image of an integral class, then $\mathcal{B}$ admits an embedding into a projective space. The line bundle $E \to \mathcal{B}$ associated to such fundamental form is, by definition, negative, and its circle bundle with the induced CR structure, strictly pseudoconvex. For any integer m let $\mathfrak{H}(\mathcal{B}, E^{\otimes m})$ be the space of holomorphic sections of $E^{\otimes m}$. The proof of Kodaira's existence theorem consists of showing that for a suitable m (a negative number here), the map sending the point $b \in \mathcal{B}$ to the kernel of the map

$$\mathfrak{H}(\mathcal{B}, E^{\otimes m}) \ni \phi \mapsto \phi(b) \in E_b^{\otimes m}$$

defines an embedding $\Psi : \mathcal{B} \to \mathbb{P}\mathfrak{H}(\mathcal{B}, E^{\otimes m})^*$. We describe an interpretation of this along the lines of the last assertion in Theorem 1.5. Fix a Hermitian metric on E and use it to induce metrics on each of the tensor powers of E. For each integer

$m \neq 0$, define $\wp_m : SE \to SE^{\otimes m}$ by

$$\wp_m(p) = \begin{cases} p \otimes \ldots \otimes p & \text{if } m > 0 \\ p^* \otimes \ldots \otimes p^* & \text{if } m < 0, \end{cases}$$

($|m|$ factors in either case) with $p^* \in E^*_{\rho(p)}$ such that $\langle p^*, p\rangle = 1$. A section ϕ of $E^{\otimes(-m)}$ is a map $E^{\otimes m} \to \mathbb{C}$ which in turn gives a map $SE \to \mathbb{C}$ by way of the formula

$$SE \ni p \mapsto f_\phi(p) = \langle \phi, \wp_m(p)\rangle \in \mathbb{C}.$$

This map has the property that

$$\frac{d}{dt} f_\phi(e^{it}p) = im f_\phi(e^{it}p).$$

Conversely, any $f : SE \to E$ with this property determines a section ϕ of $E^{\otimes(-m)}$ such that $f_\phi = f$. It is not hard to see that f_ϕ is a CR function if and only if ϕ is a holomorphic section. Suppose $E \to \mathcal{B}$ is a negative line bundle. Suppose m is so large that the map Ψ described above is an embedding. Let $\phi_1, \ldots, \phi_N$ be a basis of $\mathfrak{H}(\mathcal{B}, E^{-m})$. Then the map $F : SE \to \mathbb{C}^N$ with components f_{ϕ_j} is an equivariant CR embedding, the assertion in part (1) of Theorem 1.5. In this case, since $F(e^{it}p) = e^{imt}F(p)$, the numbers τ_j are all equal to m (here a positive number). Kodaira's embedding map consists of sending the point $b \in \mathcal{B}$ to the complex line containing $F(SE_b)$.

Theorems 1.1 and 1.5 are generalization of classical theorems about line bundles. Other generalizations of classical results about line bundles to the contexts of these theorems were given in [14] (generalizing classification by the first Chern class) and [15] (concerning a kind of Gysin sequence). We point out, however, that Theorem 1.5 applied to line bundles does not quite give Kodaira's Embedding Theorem because one cannot guarantee that the vector field $\mathcal{T}'$ alluded to in the statement about the embedding being equivariant has all τ_j equal to each other. A similar remark applies to Theorem 1.1.

The proof of Theorem (1.1), contained in Section 2, exploits an idea used by Bochner [3] to prove analytic embeddability in $\mathbb{R}^N$ of real analytic compact manifolds with analytic Riemannian metric. The rest of the paper is devoted to the proof of Theorem 1.5. In Section 3 we recall some basic facts about involutive structures and their associated complexes, including some aspects of elliptic structures (of which the subbundles $\overline{\mathcal{V}}$ in the definition of $\mathscr{F}_{\text{ell}}$ are examples). In Section 4 we discuss the complexes relevant to this work. The presentation here is motivated by earlier work on complex b-structures, see [12, 13] and [14, Section 1]. Section 5 is a preliminary analysis of the structure of the space of CR functions on $\mathcal{N}$ for a given β. This is used in Section 6 to prove that (1) $\Longrightarrow$ (2) (Proposition 6.9) and that (2) $\Longrightarrow$ (3) (Proposition 6.11) in Theorem 1.5. The implication (3) $\Longrightarrow$ (4) is proved in Section 7 (Theorem 7.1). This last section includes a result (Proposition 7.4) about a decomposition of the space of L^2 CR functions into eigenspaces of $\mathcal{L}_\mathcal{T}$. This can be interpreted as giving a global version of the Baouendi-Treves Approximation Theorem [1], see Remark 7.14. The implication (4) $\Longrightarrow$ (1) is immediate.

2. Real embeddings

Suppose $(\mathcal{N},\mathcal{T})\in\mathscr{F}$ and fix some $\mathcal{T}$-invariant Riemannian metric g on $\mathcal{N}$. Let Δ denote the Laplace-Beltrami operator. Since $\mathcal{L}_{\mathcal{T}}g=0$, Δ commutes with $\mathcal{T}$. It is of course well known that the eigenspaces of Δ are finite-dimensional and consist of smooth functions. Since Δ commutes with $\mathcal{T}$, these eigenspaces are invariant under $-i\mathcal{T}$. The latter operator acts on these finite-dimensional spaces as a selfadjoint operator (with the inner product of the L^2 space defined by the Riemannian density), in particular with real eigenvalues. Let

$$\mathcal{E}_{\tau,\lambda}=\{\phi\in C^\infty(\mathcal{N}):-i\mathcal{T}\phi=\tau\phi,\ \Delta\phi=\lambda\phi\}$$

and let

$$\operatorname{spec}(-i\mathcal{T},\Delta)=\{(\tau,\lambda):\mathcal{E}_{\tau,\lambda}\neq 0\}.$$

The latter set, the joint spectrum of Δ and $-i\mathcal{T}$, is a discrete subset of $\mathbb{R}^2$. Since Δ is a real operator (that is, $\Delta\overline{\phi}=\overline{\Delta\phi}$),

$$(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)\implies(-\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta). \tag{2.1}$$

Note that the map F satisfies $F_*\mathcal{T}=\mathcal{T}'$ with $\mathcal{T}'$ given by (1.2) if and only if its component functions f^j satisfy $\mathcal{T}f^j=i\tau_j f^j$. This justifies using functions in the spaces $\mathcal{E}_{\tau,\lambda}$ as building blocks for the components of F. For each $(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$ let $\phi_{\tau,\lambda,j}$, $j=1,\dots,N_{\tau,\lambda}$, be an orthonormal basis of $\mathcal{E}_{\tau,\lambda}$, so

$$\{\phi_{\tau,\lambda,j}:(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta),\ j=1,\dots,N_{\tau,\lambda}\}$$

is an orthonormal basis of $L^2(\mathcal{N})$. To construct F we will take advantage of the following two properties of the $\phi_{\tau,\lambda,j}$:

(1) for all $p_0\in\mathcal{N}$, $\mathbb{C}T^*_{p_0}\mathcal{N}=\operatorname{span}\{d\phi_{\tau,\lambda,j}(p_0):(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta),\ j=1,\dots,N_{\tau,\lambda}\}$;
(2) the functions $\phi_{\tau,\lambda,j}$, $(\tau,\lambda)\in\operatorname{spec}(-i\mathcal{T},\Delta)$, $j=1,\dots,N_{\tau,\lambda}$, separate points of $\mathcal{N}$.

To prove the first assertion, suppose that the span of the $d\phi_{\tau,\lambda,j}(p_0)$ is a proper subspace W of $\mathbb{C}T^*_{p_0}\mathcal{N}$, and let $f:\mathcal{N}\to\mathbb{C}$ be a smooth function such that $df(p_0)\notin W$. By standard results from the theory of elliptic selfadjoint operators on compact manifolds, the Fourier series of f,

$$f=\sum_{(\tau,\lambda)\in\Sigma}\sum_{j=1}^{N_{\tau,\lambda}}f_{\tau,\lambda,j}\phi_{\tau,\lambda,j} \tag{2.2}$$

converges to f in $C^\infty(\mathcal{N})$; here we used Σ to denote $\operatorname{spec}(-i\mathcal{T},\Delta)$. So

$$df(p_0)=\sum_{(\tau,\lambda)\in\Sigma}\sum_{j=1}^{N_{\tau,\lambda}}f_{\tau,\lambda,j}\,d\phi_{\tau,\lambda,j}(p_0)$$

with uniform convergence of the series. The terms of the series belong to W, a complete space because it is finite-dimensional, so the convergence takes place in W. But $df(p_0)\notin W$, a contradiction. Thus in fact $W=\mathbb{C}T^*_{p_0}\mathcal{N}$ as claimed.

The second assertion is proved by using the pointwise convergence of the series (2.2) for a smooth function f separating two distinct points p_0 and p_1 to contradict the supposition that $\phi_{\tau,\lambda,j}(p_0)=\phi_{\tau,\lambda,j}(p_1)$ for all values of the indices.

It follows from property (1) that there are (τ_k,λ_k,j_k), $k=1,\dots,\dim\mathcal{N}$ such that the differentials at p_0 of the functions $f^k=\phi_{\tau_k,\lambda_k,j_k}$ span $\mathbb{C}T^*_{p_0}\mathcal{N}$. Then, if

v is a real tangent vector at p_0, the condition $df^k(v) = 0$ for all k implies $v = 0$. The same property is true if some or all of the functions f^k are replaced by their conjugates. So replacing f^k by $\overline{f}^k$ if $\tau_k < 0$ we get that the map

$$p \mapsto (f^1(p), \dots, f^{\dim \mathcal{N}}(p))$$

has injective differential at p_0 (hence in a neighborhood of p_0) and components that satisfy $\mathcal{T} f^k = i\tau_k f^k$ with $\tau_k > 0$, see (2.1).

By the compactness of $\mathcal{N}$, there are smooth functions $\tilde{f}^1, \dots, \tilde{f}^{\tilde{N}}$ such that $\mathcal{T}\tilde{f}^k = i\tau_k \tilde{f}^k$ for each k with $\tau_k > 0$, and such that the map $\tilde{F} : \mathcal{N} \to \mathbb{C}^{\tilde{N}}$ with components f^k is an immersion. The origin of $\mathbb{C}^{\tilde{N}}$ is not in the image of $\tilde{F}$. Indeed, if there is p_0 such that $\tilde{f}^k(p_0) = 0$ for all k, then $\mathcal{T}\tilde{f}^k(p_0) = i\tau_k \tilde{f}(p_0) = 0$ for all k, so $\mathcal{T}(p_0)$ belongs to the kernel of $d\tilde{F}(p_0)$, a contradiction.

Since $\|\tilde{F}(p)\| \neq 0$ for all p, the map $p \mapsto \|\tilde{F}(p)\|^{-1}\tilde{F}(p)$ is smooth and has image in $S^{2\tilde{N}-1}$. However, it may not be an immersion, since the differential of the radial projection $\mathbb{C}^{\tilde{N}}\backslash 0 \to S^{2\tilde{N}-1}$ has nontrivial kernel at every point: the kernel at $z \in \mathbb{C}^{\tilde{N}}\backslash 0$ is the radial vector $R = \sum_\ell z^\ell \partial_{z^\ell} + \overline{z}^\ell \partial_{\overline{z}^\ell}$. To fix this problem we augment $\tilde{F}$ by adjoining the functions $(\tilde{f}^k)^2$: redefine $\tilde{F}$ to be

$$\tilde{F} = (\tilde{f}^1, \dots, \tilde{f}^{\tilde{N}}, (\tilde{f}^1)^2, \dots, (\tilde{f}^{\tilde{N}})^2).$$

Then $\tilde{F}$ is again an immersion. Moreover, for all $p \in \mathcal{N}$, $R(\tilde{F}(p)) \notin \operatorname{rg} d\tilde{F}(p)$. To see this, suppose $v \in T_{p_0}\mathcal{N}$ is such that

$$d\tilde{F}(v) = cR(\tilde{F}(p_0))$$

for some c. Then

$$\langle d\tilde{f}^k, v\rangle = c\tilde{f}^k(p_0) \text{ and } \langle d(\tilde{f}^k)^2, v\rangle = c(\tilde{f}^k(p_0))^2,\ k = 1, \dots, \tilde{N}.$$

Using the first set of equations in the second we get

$$c(\tilde{f}^k(p_0))^2 = \langle d(\tilde{f}^k)^2, v\rangle = 2\tilde{f}^k(p_0)\langle d\tilde{f}^k, v\rangle = 2c(\tilde{f}^k(p_0))^2 \text{ for all } k$$

so, since $\tilde{f}^k(p_0) \neq 0$ for some k, $c = 2c$, hence $c = 0$. Thus the composition of $\tilde{F}$ with the radial projection on $S^{4\tilde{N}-1}$,

$$F_0(p) = \frac{1}{\|\tilde{F}(p)\|}\tilde{F}(p),$$

is an immersion. Let $N_0 = 2\tilde{N}$ and let $f^1, \dots, f^{N_0}$ denote the components of F_0. Since $\mathcal{T}|\tilde{f}^j|^2 = 0$ (because $\mathcal{T}\tilde{f}^j = i\tau_j \tilde{f}^j$ and τ_j is real), $\mathcal{T} f^j = i\tau_j f^j$ with $\tau_j > 0$.

We will now augment F_0 so as to obtain an injective map. Let

$$Z = \{(p_0, p_1) \in \mathcal{N} \times \mathcal{N} : p_0 \neq p_1,\ f^k(p_0) = f^k(p_1) \text{ for all } k\}.$$

Since F_0 is an immersion, the diagonal in $\mathcal{N} \times \mathcal{N}$ has a neighborhood U on which the condition

$$(p_0, p_1) \in U \text{and } F_0(p_0) = F_0(p_1) \implies p_0 = p_1$$

holds. Thus Z is a closed set. Suppose $(p_0, p_1) \in Z$. By the second property of the functions $\phi_{\tau,\lambda,j}$ there is f smooth such that $\mathcal{T} f = i\tau f$ and $f(p_0) \neq f(p_1)$. If the latter happens, then also $\overline{f}(p_0) \neq \overline{f}(p_1)$, so we may assume $\tau > 0$. With such f the map

$$F_1 : p \mapsto \frac{1}{\sqrt{1 + |f(p)|^2}}(F(p), f(p)),$$

which has image in the unit sphere in $\mathbb{C}^{N_0+1}$, separates p_0 and p_1. Indeed, if

$$F(p_0)/\sqrt{1+|f(p_0)|^2} = F(p_1)/\sqrt{1+|f(p_1)|^2} \text{ and}$$
$$f(p_0)/\sqrt{1+|f(p_0)|^2} = f(p_1)/\sqrt{1+|f(p_1)|^2},$$

then, since $F(p_0) = F(p_1)$ (because $(p_0,p_1) \in Z$), $\sqrt{1+|f(p_0)|^2} = \sqrt{1+|f(p_1)|^2}$, so $f(p_0) = f(p_1)$ contradicting the choice of f. So $F_1(p_0) \neq F_1(p_1)$, and (p_0,p_1) has a neighborhood U such that $(p,p') \in U \implies F_1(p) \neq F_1(p')$. Using the compactness of Z we get a finite number of maps $F_1,\dots,F_L$, each mapping $\mathcal{N}$ into the unit sphere in $\mathbb{C}^{N_0+1}$, such that $(p_0,p_1) \in Z$ implies $F_\ell(p_0) \neq F_\ell(p_1)$ for some ℓ. Then, with $N = N_0 + (N_0+1)L + 1$,

$$F = \frac{1}{\sqrt{L+1}}(F_0, F_1, \dots, F_L) : \mathcal{N} \to S^{2N+1}$$

is an embedding whose components f^j satisfy $\mathcal{T} f^j = i\tau_j f^j$ with $\tau_j > 0$, hence $F_*\mathcal{T} = T'$ with $\mathcal{T}'$ given by (1.2) as claimed.

That no component of the map F just constructed is flat at any point of $\mathcal{N}$ is a consequence of the fact that these functions are constructed out of eigenfunctions of a second order elliptic real operator (see [7, Theorem 17.2.6]). In particular, the set $\{p \in \mathcal{N} : \forall j\ F^j(p) \neq 0\}$ is dense in $\mathcal{N}$.

Remark 2.3. The last assertion of Theorem 1.1 was an essential component in the proof of Proposition 3.7 used in [14].

3. Involutive structures

Let $\mathcal{M}$ be a smooth manifold. An involutive structure on $\mathcal{M}$ is a subbundle of the complexification $\mathbb{C}T\mathcal{M}$ of the tangent bundle of $\mathcal{M}$. We will briefly review some facts in connection with such structures here and then discuss particularities in the context of Theorem 7.1. For a detailed account of various aspects of such structures see Treves [18, 19, 20].

Associated to any involutive structure $\mathcal{W} \subset \mathbb{C}T\mathcal{M}$ there is a complex based on the exterior powers of the dual bundle:

$$\cdots \to C^\infty(\mathcal{M}; \textstyle\bigwedge^q \mathcal{W}^*) \xrightarrow{\mathfrak{D}} C^\infty(\mathcal{M}; \textstyle\bigwedge^{q+1} \mathcal{W}^*) \to \cdots. \tag{3.1}$$

Namely, if $\eta \in C^\infty(\mathcal{M}; \bigwedge^q \mathcal{W}^*)$ and $V_0,\dots,V_q$ are smooth sections of $\mathcal{W}$, then

$$\begin{aligned}(q+1)\mathfrak{D}\eta(V_0,\dots,V_q) &= \sum_j (-1)^j V_j \eta(V_0,\dots,\hat{V}_j,\dots,V_q)\\ &\quad + \sum_{j<k} (-1)^{j+k} \eta([V_j,V_k], V_1,\dots,\hat{V}_j,\dots,\hat{V}_k,\dots,V_q).\end{aligned} \tag{3.2}$$

These satisfy

$$\mathfrak{D}^2 = 0$$

and

$$\mathfrak{D}(\eta \wedge \zeta) = \mathfrak{D}(\eta) \wedge \zeta + (-1)^q \eta \wedge \mathfrak{D}(\zeta) \tag{3.3}$$

if $\eta \in C^\infty(\mathcal{M}; \bigwedge^q \mathcal{W}^*)$ and $\zeta \in C^\infty(\mathcal{M}; \bigwedge^{q'} \mathcal{W}^*)$. For a function f we have $\mathfrak{D} f = \iota^* df$, where $\iota^* : \mathbb{C}T^*\mathcal{M} \to \mathcal{W}^*$ is the dual of the inclusion homomorphism $\iota : \mathcal{W} \to \mathbb{C}T\mathcal{M}$. This just means that

$$\langle \mathfrak{D} f, v \rangle = v f \tag{3.4}$$

if $v \in \mathcal{W}$.

The structure $\mathcal{W}$ is said to be elliptic if $\mathcal{W}+\overline{\mathcal{W}} = \mathbb{C}T\mathcal{M}$, the reason being that the complex (3.1) is elliptic if and only if $\mathcal{W}$ is. If $\mathcal{W}$ is an elliptic structure, $\mathcal{W}\cap\overline{\mathcal{W}}$ is the complexification of a subbundle of $T\mathcal{M}$; its integral manifolds are called the real leaves of the structure.

Suppose $\mathcal{W}$ is an elliptic structure, By a theorem of Nirenberg [17] (a consequence of the Newlander-Nireberg Theorem [16]), every point $p_0 \in \mathcal{M}$ has a neighborhood U on which there are local coordinates

$$x^1, \dots, x^{2n}, t^1, \dots, t^\kappa$$

such that, with $z^\mu = x^\mu + ix^{\mu+n}$, $\mathcal{W}|_U$ is the span of the vector fields

$$\partial_{\overline{z}^1}, \dots, \partial_{\overline{z}^n}, \partial_{t^1}, \dots, \partial_{t^\kappa}. \tag{3.5}$$

Such a local chart $(z^1, \dots, z^n, t^1, \dots, t^\kappa)$ is called a hypoanalytic chart (Baouendi-Chang-Treves [2], Treves [20]). The intersection of the real leaves and U are the level sets of the function $p \mapsto (z^1(p), \dots, z^n(p))$. If U is connected and $\zeta : U \to \mathbb{C}$ satisfies $\mathfrak{D}\zeta = 0$, then ζ is constant on the connected components of the real leaves in U and a holomorphic function of the z^μ.

Lemma 3.6. *Suppose that $\mathcal{M}$ is connected, let $\mathcal{W} \subset \mathbb{C}T^*\mathcal{M}$ be an elliptic structure, and let $\beta \in C^\infty(\mathcal{M}; \mathcal{W}^*)$ be $\mathfrak{D}$-closed. If $\zeta : \mathcal{M} \to \mathbb{C}$ is not identically zero and $\mathfrak{D}\zeta + \zeta\beta = 0$, then the set $\{p \in \mathcal{M} : \zeta(p) = 0\}$ has empty interior.*

Proof. Let $p_0 \in \zeta^{-1}(0)$ and let $(z^1, \dots, z^n, t^1, \dots, t^\kappa)$ be a hypoanalytic chart centered at p_0, mapping its domain U onto $B \times C$ where B is a ball in $\mathbb{C}^n$ with center 0 and C is the cube $(-1,1)^\kappa \subset \mathbb{R}^\kappa$. We will show in a moment that there is $f : U \to \mathbb{C}$ such that $\mathfrak{D}f = \beta$ in U. Assuming this, we have

$$\mathfrak{D}(e^f\zeta) = e^f(\mathfrak{D}\zeta + \zeta\mathfrak{D}f) = e^f(-\zeta\beta + \zeta\beta) = 0$$

so $e^f\zeta$ is a holomorphic function of the z^μ. Thus if the set $\zeta^{-1}(0)\cap U$ does not have empty interior then ζ vanishes on U. A simple argument using the connectedness of $\mathcal{M}$ then leads to the conclusion that if the interior of $\zeta^{-1}(0)$ is not empty, then ζ is identically 0.

To complete the proof we show that β is exact on U using a well-known argument. Over U, the sections $\mathfrak{D}\,\overline{z}^\mu$, $\mathfrak{D}t^j$ of $\mathcal{W}^*$ form the frame dual to the frame (3.5) of $\mathcal{W}$. Writing

$$\beta = \sum_{\mu=1}^n \beta_\mu \mathfrak{D}\,\overline{z}^\mu + \sum_{j=1}^\kappa \beta_j \mathfrak{D}t^j$$

we have

$$\mathfrak{D}\beta = \sum_{\mu<\nu}\Big(\frac{\partial\beta_\nu}{\partial\overline{z}^\mu} - \frac{\partial\beta_\mu}{\partial\overline{z}^\nu}\Big)\mathfrak{D}\,\overline{z}^\mu \wedge \mathfrak{D}\,\overline{z}^\nu + \sum_{\mu=1}^n\sum_{j=1}^\kappa\Big(\frac{\partial\beta_j}{\partial\overline{z}^\mu} - \frac{\partial\beta_\mu}{\partial t^j}\Big)\mathfrak{D}\,\overline{z}^\mu \wedge \mathfrak{D}t^j + \sum_{j<k}\Big(\frac{\partial\beta_k}{\partial t^j} - \frac{\partial\beta_j}{\partial t^k}\Big)\mathfrak{D}t^j \wedge \mathfrak{D}t^k.$$

From the condition $\mathfrak{D}\beta = 0$ we derive the existence of a smooth function g such that $\partial g/\partial t^j = \beta_j$ for each j. Then

$$\beta' = \beta - \mathfrak{D}g = \sum_{\mu=1}^n\Big(\beta_\mu - \frac{\partial g}{\partial\overline{z}^\mu}\Big)\mathfrak{D}\,\overline{z}^\mu$$

is again $\mathfrak{D}$-closed, and consequently the coefficients of β' are independent of the t^j. We may then view β' as a $(0,1)$-form, and as such it is $\overline{\partial}$-closed. Since B is a ball, there is $h(z)$ such that $\overline{\partial} h = \beta'$, and it follows that $\beta = \mathfrak{D}(g+h)$ in U. □

We end our discussion of general elliptic structures with the following observation:

Lemma 3.7. *Suppose that $\mathcal{M}$ is compact and connected. If $\zeta : \mathcal{M} \to \mathbb{C}$ solves $\mathfrak{D}\zeta = 0$, then ζ is constant.*

Proof. Let p_0 be an extremal point of $|\zeta|$. Fix a hypoanalytic chart (z,t) for $\overline{\mathcal{V}}$ centered at p_0. Since $\mathfrak{D}\zeta = 0$, $\zeta(z,t)$ is independent of t and $\partial_{\overline{z}^\nu}\zeta = 0$. So there is a holomorphic function Z defined in a neighborhood of 0 in $\mathbb{C}^n$ such that $\zeta = Z \circ z$. Then $|Z|$ has a maximum at 0, so Z is constant near 0. Therefore ζ is constant, say $\zeta(p) = c$, near p_0. Let $C = \{p : \zeta(p) = c\}$, a closed set. Let $p_1 \in C$. Since p_1 is also an extremal point of ζ, the above argument gives that ζ is constant near p_1, therefore equal to c. Thus C is open, and consequently ζ is constant on $\mathcal{M}$. □

4. Underlying complexes

Fix $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$, that is, $(\mathcal{N},\mathcal{T}) \in \mathscr{F}$, $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ is an involutive elliptic subbundle with $\mathcal{V}\cap\overline{\mathcal{V}} = \operatorname{span}_{\mathbb{C}}\mathcal{T}$, and there is a global section $\beta \in C^\infty(\mathcal{N};\overline{\mathcal{V}}^*)$ such that

$$\text{a) } \langle \beta, \mathcal{T}\rangle = -i; \qquad \text{b) } \overline{\mathbb{D}}\beta = 0 \tag{4.1}$$

where $\overline{\mathbb{D}}$ refers to the coboundary operator of the induced differential complex on $\mathcal{V}^*$:

$$\cdots \to C^\infty(\mathcal{N};{\textstyle\bigwedge}^q\overline{\mathcal{V}}^*) \xrightarrow{\overline{\mathbb{D}}} C^\infty(\mathcal{N};{\textstyle\bigwedge}^{q+1}\overline{\mathcal{V}}^*) \to \cdots. \tag{4.2}$$

In addition to the complex (4.2), which exists independently of β, there is another complex on $\mathcal{N}$ induced by β, namely let

$$\overline{\mathcal{K}}_\beta = \{v \in \overline{\mathcal{V}} : \langle \beta, v\rangle = 0\}.$$

Indeed, $\mathcal{K}_\beta$ is involutive: For if V and W are sections of $\mathcal{K}_\beta$, then by (3.2),

$$\langle \beta, [V,W]\rangle = -2\overline{\mathbb{D}}\beta(V,W) + V\langle\beta,W\rangle - W\langle\beta,V\rangle$$

which vanishes by property b) above and because $\langle\beta,V\rangle = \langle\beta,W\rangle = 0$. Now, $\overline{\mathcal{K}}_\beta$ is a CR structure: $\overline{\mathcal{K}}_\beta\cap\mathcal{K}_\beta = 0$. To see this, suppose $v \in \overline{\mathcal{K}}_\beta\cap\mathcal{K}_\beta$. Then in particular $v \in \overline{\mathcal{V}}\cap\mathcal{V}$, so $v = c\mathcal{T}$ for some c. Thus $0 = \langle\beta,v\rangle = \langle\beta,c\mathcal{T}\rangle = ic$, hence $v = 0$. We will write $\overline{\partial}_b$ for the coboundary operators of the complex

$$\cdots \to C^\infty(\mathcal{N};{\textstyle\bigwedge}^q\overline{\mathcal{K}}^*_\beta) \to C^\infty(\mathcal{N};{\textstyle\bigwedge}^{q+1}\overline{\mathcal{K}}^*_\beta) \to \cdots.$$

Occasionally there will be two such complexes involved, determined by sections β and β'. We will not distinguish this in the notation.

There is a third complex associated with $\overline{\mathcal{V}}$ and β, in which the terms of the cochain complex are those in (4.2), but the coboundary operator is

$$\overline{\mathcal{D}}_q(\sigma)\phi = \overline{\mathbb{D}}\phi + i\sigma\beta\wedge\phi, \quad \phi \in C^\infty(\mathcal{N};{\textstyle\bigwedge}^q\overline{\mathcal{V}}^*) \tag{4.3}$$

with a fixed $\sigma \in \mathbb{C}$. That $\overline{\mathcal{D}}_{q+1}(\sigma)\overline{\mathcal{D}}_q(\sigma) = 0$ follows immediately form the corresponding property for $\overline{\mathbb{D}}$ together with b) in (4.1). This complex is, again, elliptic. Write $H^q_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N})$ for the cohomology groups and let

$$\operatorname{spec}^q(\overline{\mathcal{D}}) = \{\sigma : H^q_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N}) \neq 0\}.$$

Lemma 4.4. *The cohomology groups $H^q_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N})$ are finite-dimensional for each $\sigma \in \mathbb{C}$. For each q, the set $\operatorname{spec}^q(\overline{\mathcal{D}})$ is closed and discrete, in fact*

$$\{\sigma \in \operatorname{spec}^q(\overline{\mathcal{D}}) : -a \leq \Im\sigma \leq a\}$$

is finite for each $a > 0$.

Proof. Fix a $\mathcal{T}$-invariant metric g on $\mathcal{N}$ for which $g(\mathcal{T},\mathcal{T}) = 1$. It determines a metric on $\overline{\mathcal{V}}$, hence on the various exterior powers of $\overline{\mathcal{V}}^*$. We use these metrics and the Riemannian measure to give an L^2 inner product to each of the spaces $C^\infty(N; \bigwedge^q\overline{\mathcal{V}}^*)$. Let

$$\overline{\mathcal{D}}^\star_q(\sigma) : C^\infty(\mathcal{N}; {\textstyle\bigwedge}^{q+1}\overline{\mathcal{V}}^*) \to C^\infty(\mathcal{N}; {\textstyle\bigwedge}^{q}\overline{\mathcal{V}}^*)$$

denote the formal adjoint of $\overline{\mathcal{D}}_q(\overline{\sigma})$; it depends holomorphically on σ. Define

$$\Box_q(\sigma) = \overline{\mathcal{D}}^\star_q(\sigma)\overline{\mathcal{D}}_q(\sigma) + \overline{\mathcal{D}}_{q-1}(\sigma)\overline{\mathcal{D}}^\star_{q-1}(\sigma).$$

This is a family of elliptic operators depending holomorphically on σ. Since $\Box_q(\sigma)$ is elliptic (because the complex is) and $\mathcal{N}$ is compact, this is a Fredholm family. Furthermore, if $\boldsymbol{\sigma}(\Box_q(\sigma))$ denotes the principal symbol of $\Box_q(\sigma)$ and $\|\beta\|$ denotes the pointwise norm of β, we have that

$$\boldsymbol{\sigma}(\Box_q)(\boldsymbol{\xi}) + \sigma^2\,\|\beta\|^2 I$$

is invertible at every point $\boldsymbol{\xi} \in T^*\mathcal{N}$ and every $\sigma \in \mathbb{C}$ with estimates

$$\left\| \big(\boldsymbol{\sigma}(\Box_q)(\boldsymbol{\xi}) + \sigma^2\,\|\beta\|^2 I\big)^{-1} \right\| \leq \frac{C}{\|\boldsymbol{\xi}\|^2 + |\sigma|^2}$$

with uniform C for arbitrary $\boldsymbol{\xi}$ and σ such that $|\Im\sigma| \leq a$ (C depends on a) because $\|\beta\|$ is nowhere zero. This estimate implies that for each $a > 0$ there is b such that $\Box_q(\sigma)$ is invertible if $|\Im\sigma| \leq a$ and $|\Re\sigma| > b$. Since $\Box_q(\sigma)$ is a holomorphic Fredholm family, the intersection of

$$\Sigma_q = \{\sigma \in \mathbb{C} : \Box_q(\sigma) \text{ is invertible}\}$$

with any horizontal strip $\{\sigma \in \mathbb{C} : |\Im\sigma| \leq a\}$ is finite. We now show that the analogous statement holds for $\operatorname{spec}^q(\overline{\mathcal{D}})$. Let

$$\mathcal{G}_q(\sigma) : L^2(\mathcal{N}; E^q_{\mathcal{N}}) \to H^2(\mathcal{N}; E^q_{\mathcal{N}})$$

be the inverse of $\Box_q(\sigma)$, $\sigma \notin \Sigma_q$. The map $\sigma \mapsto \mathcal{G}_q(\sigma)$ is meromorphic with poles in Σ_q. The operators $\Box_q(\sigma)$ are the Laplacians of the complex (4.2) with the coboundary operators (4.3) when σ is real. Thus for $\sigma \in \mathbb{R}\backslash(\operatorname{spec}_{b,\mathcal{N}}(\Box_q) \cup \operatorname{spec}_{b,\mathcal{N}}(\Box_{q+1}))$ we have

$$\overline{\mathcal{D}}_q(\sigma)\mathcal{G}_q(\sigma) = \mathcal{G}_{q+1}(\sigma)\overline{\mathcal{D}}_q(\sigma), \quad \overline{\mathcal{D}}_q(\sigma)^\star\mathcal{G}_{q+1}(\sigma) = \mathcal{G}_q(\sigma)\overline{\mathcal{D}}_q(\overline{\sigma})^\star$$

by standard Hodge theory. Since all operators depend holomorphically on σ, the same equalities hold for $\sigma \in \mathfrak{R} = \mathbb{C}\backslash(\Sigma_q \cup \Sigma_{q+1})$. It follows that

$$\overline{\mathcal{D}}^\star_q(\sigma)\overline{\mathcal{D}}_q(\sigma)\mathcal{G}_q(\sigma) = \mathcal{G}_q(\sigma)\overline{\mathcal{D}}^\star_q(\sigma)\overline{\mathcal{D}}_q(\sigma)$$

in $\mathfrak{R}$. By analytic continuation the equality holds on all of $\mathbb{C}\backslash\Sigma_q$. Thus if $\sigma_0 \notin \Sigma_q$ and ϕ is a $\overline{\mathcal{D}}_q(\sigma_0)$-closed section, $\overline{\mathcal{D}}_q(\sigma_0)\phi = 0$, then the formula

$$\phi = [\overline{\mathcal{D}}_q^\star(\sigma_0)\overline{\mathcal{D}}_q(\sigma_0) + \overline{\mathcal{D}}_{q-1}(\sigma_0)\overline{\mathcal{D}}_{q-1}^\star(\sigma_0)]\mathcal{G}_q(\sigma_0)\phi$$

leads to

$$\phi = \overline{\mathcal{D}}_{q-1}(\sigma_0)[\overline{\mathcal{D}}_{q-1}^\star(\sigma_0)\mathcal{G}_q(\sigma_0)\phi].$$

Therefore $\sigma_0 \notin \operatorname{spec}^q(\overline{\mathcal{D}})$. Thus $\operatorname{spec}^q(\overline{\mathcal{D}}) \subset \Sigma_q$. ☐

Remark 4.5. The argument concerning the poles of the inverse of $\Box_q(\sigma)$ is extracted from a related problem in the analysis of elliptic operators on b-manifolds, see Melrose [11]

Later we will allow replacing the section β by an equivalent section in the following sense.

Definition 4.6. Two smooth sections β, β' of $\overline{\mathcal{V}}^*$ satisfying (4.1) are equivalent if $\beta' - \beta = \overline{\mathbb{D}}u$ for some real-valued function u. The class of β is denoted $\boldsymbol{\beta}$.

Lemma 4.7. *Suppose β, β' are equivalent, let $\overline{\mathcal{D}}(\sigma)$, $\overline{\mathcal{D}}'(\sigma)$ be the associated operators. Then*

$$H^q_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N}) \approx H^q_{\overline{\mathcal{D}}'(\sigma)}(\mathcal{N})$$

for all q and σ. Consequently $\operatorname{spec}^q(\overline{\mathcal{D}})$ *depends only on the class of β.*

Proof. There is u real valued such that $\beta' = \beta + \overline{\mathbb{D}}u$. Using (3.3) and $\overline{\mathbb{D}}e^{i\sigma u} = e^{i\sigma u}\overline{\mathbb{D}}u$ we see that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C^\infty(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*) & \xrightarrow{\overline{\mathcal{D}}_\beta(\sigma)} & C^\infty(\mathcal{N};\bigwedge^{q+1}\overline{\mathcal{V}}^*) & \longrightarrow & \cdots \\ & & {\scriptstyle e^{i\sigma u}}\big\downarrow & & {\scriptstyle e^{i\sigma u}}\big\downarrow & & \\ \cdots & \longrightarrow & C^\infty(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*) & \xrightarrow{\overline{\mathcal{D}}_{\beta'}(\sigma)} & C^\infty(\mathcal{N};\bigwedge^{q+1}\overline{\mathcal{V}}^*) & \longrightarrow & \cdots \end{array} \tag{4.8}$$

is a cochain isomorphism for any σ. ☐

5. CR functions

We continue our discussion with a fixed element $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ of $\mathscr{F}_{\text{ell}}$ and section β of $\overline{\mathcal{V}}^*$ satisfying (4.1). The section β gives a CR structure $\overline{\mathcal{K}}_\beta = \ker\beta$ and operators $\overline{\mathcal{D}}(\sigma)$ defined in (4.3).

The one-parameter group of diffeomorphisms generated by $\mathcal{T}$ will be denoted by $t \mapsto \mathfrak{a}_t$. We write $\mathcal{O}_p$ for the orbit of $\mathcal{T}$ through p. The integral curves of $\mathcal{T}$ need not be periodic, i.e., the orbits need not be closed.

Lemma 5.1. *A distribution $\zeta \in C^{-\infty}(\mathcal{N})$ solves $\overline{\mathcal{D}}(\sigma)\zeta = 0$ if and only if it is a CR function and satisfies*

$$\mathcal{T}\zeta + \sigma\zeta = 0 \tag{5.2}$$

If $\overline{\mathcal{D}}(\sigma)\zeta = 0$, then ζ is smooth.

Proof. Since $\overline{\mathcal{V}} = \overline{\mathcal{K}}_\beta \oplus \operatorname{span}\mathcal{T}$, the statement that $\overline{\mathbb{D}}\zeta + i\sigma\beta\zeta$ vanishes is equivalent to

$$\langle\overline{\mathbb{D}}\zeta + i\sigma\zeta\beta, v\rangle = 0 \ \forall v \in \overline{\mathcal{K}}_\beta \quad \text{and} \quad \langle\overline{\mathbb{D}}\zeta + i\sigma\beta\zeta, \mathcal{T}\rangle = 0.$$

In view of Part a) of (4.1), and since $\langle\beta, v\rangle = 0$ and $\langle\overline{\mathbb{D}}\zeta, v\rangle = \langle\overline{\partial}_b\zeta, v\rangle$ if $v \in \overline{\mathcal{K}}_\beta$, these statements are equivalent, respectively, to

$$\overline{\partial}_b\zeta = 0 \quad \text{and} \quad \mathcal{T}\zeta + \sigma\zeta = 0$$

as claimed. That ζ is smooth if $\overline{\mathcal{D}}(\sigma)\zeta = 0$ is a consequence of the complex (4.2) being elliptic (the principal symbol of $\overline{\mathcal{D}}(\sigma)$ on functions is injective). □

The space of smooth CR functions, $C^\infty_{\mathrm{CR}}(\mathcal{N}) = C^\infty(\mathcal{N}) \cap \ker\overline{\partial}_b$, is a ring. We will see that $C^\infty_{\mathrm{CR}}(\mathcal{N})$ decomposes as a direct sum of the spaces $\ker\overline{\mathcal{D}}_0(\sigma)$, $\sigma \in \operatorname{spec}^0(\overline{\mathcal{D}})$.

Lemma 5.3. *The set* $\operatorname{spec}^0(\overline{\mathcal{D}}) \subset \mathbb{C}$ *is a subset of the imaginary axis and an additive discrete semigroup with identity. If* $\operatorname{spec}^0(\overline{\mathcal{D}})$ *is not a group, then* $\operatorname{spec}^0(\overline{\mathcal{D}})\backslash 0$ *is contained in a single component of* $i\mathbb{R}\backslash\{0\}$.

Proof. That $\operatorname{spec}^0(\overline{\mathcal{D}})$ is discrete is a consequence of Lemma 4.4. Suppose $\sigma_0 \in \operatorname{spec}^0(\overline{\mathcal{D}})$ is not zero and let ζ be a nonzero function that satisfies $\overline{\mathcal{D}}(\sigma_0)\zeta = 0$; such ζ exists precisely because $\sigma_0 \in \operatorname{spec}^0(\overline{\mathcal{D}})$. Furthermore, ζ is bounded because it is smooth and $\mathcal{N}$ is compact. By Lemma 5.1, $\zeta(\mathfrak{a}_t p) = e^{-\sigma_0 t}\zeta(p)$. So $|e^{-\sigma_0 t}\zeta(p)|$ is bounded. Since ζ is not identically zero, there is p such that $\zeta(p) \neq 0$. Thus $|e^{-\sigma_0 t}|$ is bounded, hence $\Re\sigma_0 = 0$.

Since $\overline{\mathbb{D}}1 = 0$, $0 \in \operatorname{spec}^0(\overline{\mathcal{D}})$. Let σ_1, $\sigma_2 \in \operatorname{spec}^0(\overline{\mathcal{D}})$, and pick nonvanishing elements $\zeta^1 \in H^0_{\overline{\mathcal{D}}(\sigma_1)}(\mathcal{N})$, $\zeta^2 \in H^0_{\overline{\mathcal{D}}(\sigma_2)}(\mathcal{N})$. Since

$$\overline{\mathbb{D}}(\zeta^1\zeta^2) = \zeta^2\overline{\mathbb{D}}\zeta^1 + \zeta^1\overline{\mathbb{D}}(\zeta^2) = -i(\sigma_1 + \sigma_2)\zeta^1\zeta^2\beta,$$

$\zeta^1\zeta^2 \in H^0_{\overline{\mathcal{D}}(\sigma_1+\sigma_2)}(\mathcal{N})$ which by Lemma 3.6 is not identically 0 (since neither of ζ^1, ζ^2 is). Thus $\sigma_1 + \sigma_2 \in \operatorname{spec}^0(\overline{\mathcal{D}})$.

Suppose now that $\operatorname{spec}^0(\overline{\mathcal{D}})$ has elements in both components of $\mathbb{C}\backslash\mathbb{R}$ and let σ_+ be the element with smallest modulus among the elements of $\operatorname{spec}^0(\overline{\mathcal{D}})$ with positive imaginary part, and let σ_- be the analogous element with negative imaginary part. If $\sigma = \sigma_+ + \sigma_- \neq 0$, then either $\Im\sigma > 0$ and $|\sigma| < |\sigma_+|$ or $\Im\sigma < 0$ and $|\sigma| < |\sigma_-|$. Either way we get a contradiction, since $\sigma \in \operatorname{spec}^0(\overline{\mathcal{D}})$. So $\sigma_- = -\sigma_+$. In particular, $m\sigma_+ \in \operatorname{spec}^0(\overline{\mathcal{D}})$ for every $m \in \mathbb{Z}$. If $\sigma \in \operatorname{spec}^0(\overline{\mathcal{D}})$ is arbitrary, then there is $m \in \mathbb{Z}$ such that $|\sigma - m\sigma_+| < |\sigma_+|$. Consequently $\sigma - m\sigma_+ = 0$. Thus $\operatorname{spec}^0(\overline{\mathcal{D}}) = \sigma_+\mathbb{Z}$, a group. Therefore, if $\operatorname{spec}^0(\overline{\mathcal{D}})$ is not a group, then $\operatorname{spec}^0(\overline{\mathcal{D}})\backslash 0$ is contained in a single component of $\mathbb{C}\backslash\mathbb{R}$. □

Thus the space

$$\bigoplus_{\sigma\in\operatorname{spec}^0(\overline{\mathcal{D}})} H^0_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N}),$$

is a subring of $C^\infty_{\mathrm{CR}}(\mathcal{N})$ graded by $\operatorname{spec}^0(\overline{\mathcal{D}})$.

The spaces $H^0_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N})$ are particularly simple when $\operatorname{spec}^0(\overline{\mathcal{D}})$ is a group.

Proposition 5.4. *Suppose that* $\operatorname{spec}^0(\overline{\mathcal{D}})$ *is a group. Then all cohomology groups* $H^0_{\overline{\mathcal{D}}(\sigma)}(\mathcal{N})$*,* $\sigma \in \operatorname{spec}^0(\overline{\mathcal{D}})$*, are one-dimensional, and all their nonzero elements are nowhere vanishing functions.*

Proof. The dimension of $H^0_{\overline{\mathcal{D}}(0)}(\mathcal{N}) = H^0_{\overline{\mathbb{D}}}(\mathcal{N})$ is 1, since this space contains the constant functions, and only the constant functions by Lemma 3.7. If $\operatorname{spec}^0(\overline{\mathcal{D}}) = \{0\}$ there is nothing more to prove. So suppose $\operatorname{spec}^0(\overline{\mathcal{D}}) \neq \{0\}$. Pick a generator

σ_1 of $\operatorname{spec}^0(\overline{\mathcal{D}})$ and a nonzero element $\eta \in H^0_{\overline{\mathcal{D}}(\sigma_1)}(\mathcal{N})$. If $\eta' \in H^0_{\overline{\mathcal{D}}(-\sigma_1)}(\mathcal{N})$ is a nonzero element, then $\eta\eta'$ is not identically zero (Lemma 3.6) and belongs to $H^0_{\overline{\mathcal{D}}(0)}(\mathcal{N})$. Therefore $\eta\eta'$ is a nonzero constant. Thus η vanishes nowhere and η^k belongs to $H^0_{\overline{\mathcal{D}}(k\sigma_1)}(\mathcal{N})$ for each $k \in \mathbb{Z}$. If $\zeta \in H^0_{\overline{\mathcal{D}}(k\sigma_1)}(\mathcal{N})$, then $\zeta\eta^{-k}$ is a constant c, so $\zeta = c\eta^k$. Thus each group $H^0_{\overline{\mathcal{D}}(k\sigma_1)}(\mathcal{N})$ is one-dimensional and its nonzero elements vanish nowhere. □

As a consequence of the proof we get

Corollary 5.5. *Suppose that* $\operatorname{spec}^0(\overline{\mathcal{D}})$ *is a group. If* $\zeta^j \in H^0_{\overline{\mathcal{D}}(\sigma_j)}(\mathcal{N})$, $j = 1, 2$, *then* $d\zeta^1$ *and* $d\zeta^2$ *are everywhere linearly dependent.*

If $\dim \mathcal{N} = 1$, then $\mathcal{N}$ is a circle and $\operatorname{spec}^0(\overline{\mathcal{D}})$ is a group. Somewhat less trivially:

Example 5.6. Let $\mathcal{B}$ be a compact complex manifold, let $E \to \mathcal{B}$ be a flat line bundle; the holomorphic structure is the one for which the local flat section are holomorphic. Pick a hermitian metric and let $\mathcal{N}$ be the circle bundle, with the usual structure as in Example 1.7. If some power E^m, $m \neq 0$, is holomorphically trivial, then with the smallest such power, m_0, we get that $\operatorname{spec}^0(\overline{\mathcal{D}}) = im_0\mathbb{Z}$. If no such m exists, then $\operatorname{spec}^0(\overline{\mathcal{D}}) = \{0\}$.

6. CR maps into $\mathbb{C}^N$

We now analyze maps $\mathcal{N} \to \mathbb{C}^N \backslash 0$.

Proposition 6.1. *Suppose that there is a map* $F : \mathcal{N} \to \mathbb{C}^N\backslash 0$ *whose components* ζ^j *satisfy*

$$\overline{\mathbb{D}}\zeta^j + i\sigma_j\zeta^j\beta = 0 \tag{6.2}$$

with all the σ_j *in one component of* $i\mathbb{R}\backslash\{0\}$. *Then there is* $u : \mathcal{N} \to \mathbb{R}$ *smooth such that the map* $\tilde{F} : \mathcal{N} \to \mathbb{C}^N\backslash 0$ *with components* $\tilde{\zeta}^j = e^{-i\sigma_j u}\zeta^j$ *has image in* S^{2N-1}.

Proof. Let $s_j = -\Im\sigma_j$ and define $g : \mathbb{R}_+ \times (\overline{\mathbb{R}}^N_+\backslash 0) \to \mathbb{R}$ by

$$g(\rho, y^1, \dots, y^N) = \sum_{j=1}^N \rho^{-2s_j} y^j.$$

Since all s_j have the same sign, $\partial_\rho g(\rho, y)$ does not vanish. If the s_j are positive, then for fixed y, $g(\rho, y) \to \infty$ as $\rho \to 0$ and $g(\rho, y) \to 0$ as $\rho \to \infty$. An analogous statement holds if all s_j are negative. So for each $y \in \overline{\mathbb{R}}^N_+\backslash 0$ there is a unique positive $\rho(y)$ such that $g(\rho(y), y) = 1$, and $\rho(y)$ depends smoothly on y. Define $f : \mathcal{N} \to \mathbb{R}$ by $f = \rho(|\zeta^1|^2, \dots, |\zeta^N|^2)$. The function f is well defined because the ζ^j do not vanish simultaneously, is positive everywhere, and satisfies

$$\sum_{j=1}^N f^{-2s_j}|\zeta^j|^2 = 1. \tag{6.3}$$

By Lemma 5.3, $\sigma_j = is_j$ for some real number s_j. The identity $\mathcal{T}\zeta^j = -\sigma_j\zeta^j$ gives $\zeta^j(\mathfrak{a}_t p) = e^{-is_jt}\zeta^j(p)$ so

$$\mathcal{T}|\zeta^j|^2 = 0. \tag{6.4}$$

Applying $\mathcal{T}$ to both members of (6.3) and using (6.4) gives

$$-2f^{-1}\sum_{j=1}^{N} s_j f^{-2s_j}|\zeta^j|^2\mathcal{T}f = 0.$$

The function $\sum_{j=1}^{N} s_j f^{-2s_j}|\zeta^j|^2$ vanishes nowhere, since all the s_j have the same sign. Thus $\mathcal{T}f = 0$, and with $u = \log f$ we also have

$$\mathcal{T}u = 0. \tag{6.5}$$

Define $\tilde{\zeta}^j = e^{-i\sigma_j u}\zeta^j$. Then $|\tilde{\zeta}^j|^2 = f^{-2s_j}|\zeta^j|^2$, so by (6.3) the map $\tilde{F}: \mathcal{N} \to \mathbb{C}^N\backslash 0$ with components $\tilde{\zeta}^j$ has image in S^{2N-1}. □

With the notation of the proof let $\beta' = \beta + \overline{\mathbb{D}}u$. Thus $\overline{\mathbb{D}}\beta' = 0$ and because of (6.5), also $\langle \beta', \mathcal{T}\rangle = -i$. Thus β' is an admissible section of $\overline{\mathcal{V}}^*$. The functions $\tilde{\zeta}^j$ satisfy

$$\overline{\mathbb{D}}\tilde{\zeta}^j + i\sigma_j\tilde{\zeta}^j\beta' = 0. \tag{6.6}$$

Therefore, by Lemma 5.1, they are CR functions with respect to the CR structure $\overline{\mathcal{K}}_{\beta'}$. Thus $\tilde{F}: \mathcal{N} \to S^{2N-1}$ is a CR map (as was F but for the CR structure defined by β).

Using (6.5) in (6.6) gives

$$\mathcal{T}\tilde{\zeta}^j - i\tau_j\tilde{\zeta}^j = 0$$

with $\tau_j = -s_j$ (they all have the same sign). Therefore

$$\tilde{F}_*\mathcal{T}(p) = \mathcal{T}'(\tilde{F}(p))$$

where

$$\mathcal{T}' = i\sum_{j=1}^{N}\tau_j(w^j\partial_{w^j} - \overline{w}^j\partial_{\overline{w}^j}) \tag{6.7}$$

using $w^1, \dots, w^N$ as coordinates in $\mathbb{C}^N$.

Suppose that ζ vanishes nowhere and solves $\overline{\mathbb{D}}\zeta + i\sigma_0\zeta\beta$ with $\sigma_0 \neq 0$. Applying Proposition 6.1 we may assume that $|\zeta| = 1$ after a suitable change of β (in this case this just means that ζ is replaced by $\zeta/|\zeta|$ and β is changed accordingly). The following is analogous to the situation of the circle bundle of a flat line bundle, see Example 5.6.

Proposition 6.8. *Suppose $\zeta: \mathcal{N} \to S^1$ solves $\overline{\mathbb{D}}\zeta + i\sigma_0\zeta\beta = 0$ with $\sigma_0 \neq 0$. Then ζ is a submersion whose fibers are complex manifolds.*

Proof. Since ζ vanishes nowhere and $\sigma_0 \neq 0$, $\mathcal{T}\zeta \neq 0$ everywhere. Thus ζ is a submersion. Since ζ is CR with respect to $\overline{\mathcal{K}}_\beta$, $v\zeta = 0$ if $v \in \overline{\mathcal{K}}_\beta$. Since ζ is nowhere 0, also $v(1/\zeta) = 0$ if $v \in \overline{\mathcal{K}}_\beta$. But $1/\zeta = \overline{\zeta}$. Thus v is tangent to the fibers of ζ: the CR structure $\overline{\mathcal{K}}_{\beta'}$ is tangent to the fibers of ζ, and can be viewed as the $(0,1)$-tangent bundle of a complex structure. □

The case $\dim\mathcal{N} = 1$ is trivially included in Proposition 6.8. On the other hand, we have

Proposition 6.9. *Suppose that $\dim\mathcal{N} = 2n+1 > 1$ and that $F: \mathcal{N} \to \mathbb{C}^N$ is a map whose components ζ^j satisfy (6.2) with $\sigma_j \neq 0$. Suppose further that at every $p \in \mathcal{N}$, $n+1$ of the differentials $d\zeta^j(p)$ are independent. Then*

(1) $\operatorname{spec}^0(\overline{\mathcal{D}})\backslash 0$ *is contained in one component of* $i\mathbb{R}\backslash\{0\}$*;*

(2) 0 *is not in the image of* F*.*

Let $\tilde{F}:\mathcal{N}\to S^{2N-1}$ *be the map in Proposition 6.1. Then*

(3) for each p*,* $n+1$ *of the differentials* $d\tilde{\zeta}^j(p)$ *are independent.*

Proof. Since $\dim\operatorname{span} d\zeta^j > 1$, Corollary 5.5 gives that $\operatorname{spec}^0(\overline{\mathcal{D}})$ is not a group, so $\operatorname{spec}^0(\overline{\mathcal{D}})\backslash 0$ is contained in one component of $i\mathbb{R}\backslash\{0\}$ by Lemma 5.3.

To show that the image of F does not contain 0 we show that for every $p\in\mathcal{N}$ there is j_0 such that $\mathcal{T}\zeta^{j_0}(p)\neq 0$. Since $\mathcal{T}\zeta^{j_0} = -\sigma_{j_0}\zeta^{j_0}(p)$ and $\sigma_{j_0}\neq 0$, we conclude from $\mathcal{T}\zeta^{j_0}(p)\neq 0$ that $\zeta^{j_0}(p)\neq 0$.

Let then $p\in\mathcal{N}$ and suppose that the differentials $d\zeta^j(p)$, $j=1,\dots,n+1$, are independent. The restrictions to the fiber $\overline{\mathcal{K}}_{\beta,p}$ of these differentials vanish, so they give $n+1$ independent linear functions on the $(n+1)$-dimensional vector space $\mathbb{C}T_p\mathcal{N}/\overline{\mathcal{K}}_{\beta,p}$. The image of $\mathcal{T}(p)$ in this quotient is not 0, so for some j_0, $\mathcal{T}\zeta^{j_0}(p)\neq 0$. Thus the image of F does not contain 0. This and the fact that the σ_j lie in one component of $i\mathbb{R}\backslash\{0\}$ allow us to apply Proposition 6.1.

Let then $u:\mathcal{N}\to\mathbb{R}$ be the function in Proposition 6.1. Suppose again that $d\zeta^1,\dots,d\zeta^{n+1}$ are independent at p and $\mathcal{T}\zeta^{n+1}(p)\neq 0$. Let $\tilde{\zeta}^j = e^{-i\sigma_j u}\zeta^j$. Then also $\mathcal{T}\tilde{\zeta}^{n+1}(p)\neq 0$. The 1-forms

$$d\zeta^j - \frac{\mathcal{T}\zeta^j}{\mathcal{T}\zeta^{n+1}}d\zeta^{n+1},\ j=1,\dots n$$

are independent at p and a brief calculation gives that

$$d\tilde{\zeta}^j - \frac{\mathcal{T}\tilde{\zeta}^j}{\mathcal{T}\tilde{\zeta}^{n+1}}d\tilde{\zeta}^{n+1} = e^{-i\sigma_j u}\Big(d\zeta^j - \frac{\mathcal{T}\zeta^j}{\mathcal{T}\zeta^{n+1}}d\zeta^{n+1}\Big),\ j=1,\dots n \tag{6.10}$$

so these n differential forms are also independent at p. They all vanish when paired with $\mathcal{T}$. Furthermore, since $\tilde{\zeta}^{n+1}(p)\neq 0$, $\mathcal{T}\tilde{\zeta}^{n+1}(p)\neq 0$. So the differential forms (6.10) together with $d\tilde{\zeta}^{n+1}$ are independent at p. □

The differentials of the component functions of both F and $\tilde{F}$ are independent over $\mathbb{C}$. Since they are CR function, this is equivalent to F and $\tilde{F}$ being immersions.

The following result is similar to the statement in complex geometry asserting that very ample holomorphic line bundles are ample.

Proposition 6.11. *Let* $F:\mathcal{N}\to\mathbb{C}^N$ *be an immersion with image in* S^{2N-1}*,* $N>1$*, and components* ζ^j *that satisfy* (6.2)*. Then the Levi form of the CR structure* $\overline{\mathcal{K}}_\beta$ *is definite.*

Proof. Let $\theta_\beta\in C^\infty(\mathcal{N};T^*\mathcal{N})$ be the 1-form which vanishes on $\mathcal{K}_\beta\oplus\overline{\mathcal{K}}_\beta$ and satisfies $\langle\theta_\beta,\mathcal{T}\rangle = 1$. Define the Levi form with respect to θ_β as

$$\operatorname{Levi}_{\theta_\beta}(v,w) = -id\theta_\beta(v,\overline{w}),\quad v,\ w\in\mathcal{K}_{\beta,p},\ p\in\mathcal{N}.$$

In this definition we switched to the conjugate of $\overline{\mathcal{K}}_\beta$ to adapt to the traditional setup. Give S^{2N-1} the standard CR structure $\overline{\mathcal{K}}$ as in Example 1.6, let $\mathcal{T}'$ be the vector field in (6.7) and let θ' be real 1-form which vanishes on $\overline{\mathcal{K}}$ and satisfies $\langle\theta',\mathcal{T}'\rangle = 1$. Then $F^*\theta' = \theta_\beta$, since F is a CR map and $F_*\mathcal{T} = \mathcal{T}'$. The Levi form $\operatorname{Levi}_{\theta'}$ is positive (negative) definite if the τ_j are positive (negative). Let $v,\ w\in\mathcal{K}_{\beta,p}$. Then $-id\theta_\beta(v,\overline{w}) = -id\theta'(F_*v,\overline{F_*w})$. Since F is an immersion, $(v,w)\mapsto -id\theta'(F_*v,\overline{F_*w})$ is nondegenerate with the same signature as $\operatorname{Levi}_{\theta'}$. □

Propositions 6.9 and 6.11 give (1) $\implies$ (2) and (2) $\implies$ (3) in Theorem 1.5.

7. CR Embeddings

Boutet de Monvel [4] showed that if $\mathcal{N}$ is a compact strictly pseudoconvex CR manifold of dimension ≥ 5 then there is a CR embedding $F : \mathcal{N} \to \mathbb{C}^N$ for some N. The proof of the following theorem, a version of the assertion that ample line bundles are very ample, takes advantage of this and, as mentioned already, an idea of Bochner [3].

Theorem 7.1. *Suppose that $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathscr{F}_{\mathrm{ell}}$ with $\dim \mathcal{N} \geq 5$ and that β is a smooth $\overline{\mathbb{D}}$-closed section of $\overline{\mathcal{V}}^*$ such that $\overline{\mathcal{K}}_\beta$ has definite Levi form. Then there is $\beta' \in \boldsymbol{\beta}$ (see Definition 4.6) and a CR embedding $F : \mathcal{N} \to S^{2N-1} \subset \mathbb{C}^N$ of $\mathcal{N}$ with the CR structure $\overline{\mathcal{K}}_{\beta'}$ such that, with $w^1, \dots, w^N$ denoting the standard coordinates in $\mathbb{C}^N$*

$$F_* \mathcal{T} = i \sum_j \tau_j (w^j \partial_{w^j} - \overline{w}^j \partial_{\overline{w}^j})$$

for some numbers τ_j, $j = 1, \dots N$. The τ_j are all positive or all negative depending on the signature of $\mathrm{Levi}_{\theta_\beta}$.

Let $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$ be the subspace of $L^2(\mathcal{N})$ consisting of CR functions. If the Levi form of $\mathcal{K}_\beta$ is definite, as in the theorem, the space $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N})$ is infinite dimensional. Boutet de Monvel's proof of his embedding theorem consists essentially on proving that

(a) for all $p_0 \in \mathcal{N}$, $\operatorname{span}\{df(p_0) : f \in \mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N})\}$ is the annihilator of $\overline{\mathcal{K}}_\beta$ in $\mathbb{C}T^*_{p_0}\mathcal{N}$;
(b) the functions in $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N})$ separate points of $\mathcal{N}$.

The embedding map is then constructed taking advantage of these properties. In the present case we also wish (7.1) to hold, so in addition the component functions ζ^j of F should satisfy $\mathcal{L}_\mathcal{T} \zeta^j = i\tau_j \zeta_j$ with all τ_j of the same sign. We will therefore prepare for the proof of Theorem 7.1 by exhibiting a decomposition of $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$, more generally without assumptions on the Levi form, a decomposition of the $\overline{\partial}_b$ cohomology spaces in any degree, into eigenspaces of $-i\mathcal{T}$.

We begin with the following two lemmas whose proofs are elementary.

Lemma 7.2. *If α is a smooth section of the annihilator of $\overline{\mathcal{V}}$ in $\mathbb{C}T^*\mathcal{N}$, then $(\mathcal{L}_\mathcal{T}\alpha)|_{\overline{\mathcal{V}}} = 0$. Consequently, for each $p \in \mathcal{N}$ and $t \in \mathbb{R}$, $d\mathfrak{a}_t : \mathbb{C}T_p\mathcal{N} \to \mathbb{C}T_{\mathfrak{a}_t(p)}\mathcal{N}$ maps $\overline{\mathcal{V}}_p$ onto $\overline{\mathcal{V}}_{\mathfrak{a}_t(p)}$.*

It follows that there is a well defined smooth bundle homomorphism $\mathfrak{a}_t^* : \bigwedge^q \overline{\mathcal{V}}^* \to \bigwedge^q \overline{\mathcal{V}}^*$ covering $\mathfrak{a}_{-t}$. In particular, one can define the Lie derivative $\mathcal{L}_\mathcal{T}\phi$ with respect to $\mathcal{T}$ of an element in $\phi \in C^\infty(\mathcal{N}; \bigwedge^q \overline{\mathcal{V}}^*)$. The usual formula holds:

Lemma 7.3. *If $\phi \in C^\infty(\mathcal{N}; \bigwedge^q \overline{\mathcal{V}}^*)$, then $\mathcal{L}_\mathcal{T}\phi = \mathbf{i}_\mathcal{T}\overline{\mathbb{D}}\phi + \overline{\mathbb{D}}\mathbf{i}_\mathcal{T}\phi$, where $\mathbf{i}_\mathcal{T}$ denotes interior multiplication by $\mathcal{T}$. Consequently, for each t and $\phi \in C^\infty(\mathcal{N}; \bigwedge^q \overline{\mathcal{V}}^*)$, $\overline{\mathbb{D}}\mathfrak{a}_t^*\phi = \mathfrak{a}_t^*\overline{\mathbb{D}}\phi$.*

In particular, it follows from (4.1) that $\mathcal{L}_\mathcal{T}\beta = 0$. Let θ_β and the Levi form $\mathrm{Levi}_{\theta_\beta}$ be defined as at the beginning of the proof of Proposition 6.11. If $\mathrm{Levi}_{\theta_\beta}$ is either positive or negative definite (as in the hypothesis of Theorem 7.1) we may use it to define a Hermitian metric on $\overline{\mathcal{K}}_\beta$ and extend it to $\overline{\mathcal{V}}$ so that $\mathcal{T}$ is a unit vector field orthogonal to $\overline{\mathcal{K}}_\beta$. Lemma 7.3 gives that $\mathcal{L}_\mathcal{T}\beta = 0$, so $\overline{\mathcal{K}}_\beta$, θ, hence also $\mathrm{Levi}_{\theta_\beta}$ are all $\mathcal{T}$-invariant, h is $\mathcal{T}$-invariant. This metric gives an obvious metric on $\mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta \oplus \operatorname{span}_\mathbb{C} \mathcal{T}$ which in turn gives a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$ giving a $\mathcal{T}$-invariant positive density on $\mathcal{N}$.

In the general case where there is no assumption on the behavior of $\mathrm{Levi}_{\theta_\beta}$ we first construct a $\mathcal{T}$-invariant Hermitian metric on $\overline{\mathcal{K}}_\beta$ as follows. Fix some $\mathcal{T}$-invariant metric $\tilde{g}$ on $\mathcal{N}$, let $\mathcal{H} = (\mathcal{K}_\beta + \overline{\mathcal{K}}_\beta) \cap T\mathcal{N}$ and define

$$g(v,w) = \frac{1}{2}(\tilde{g}(u,v) + \tilde{g}(Ju,Jv)), \quad u,v \in \mathcal{H}_p,\ p \in \mathcal{N}$$

where J is the complex structure on $\mathcal{H}$ for which the $(0,1)$ subbundle of $\mathbb{C}\mathcal{H}$ is $\overline{\mathcal{K}}_\beta$. Since $g(Ju,Jv) = g(u,v)$, there is an induced hermitian metric h on $\overline{\mathcal{K}}_\beta$. Now define the rest of the object as was done in the previous paragraph.

Use the metric h (extended to each exterior power $\bigwedge^q\overline{\mathcal{V}}^*$) and the Riemannian density to define $L^2(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*)$ and the formal adjoint operators $\overline{\partial}_b^\star$. With these, construct the Laplacian $\Box_{b,q}$ in each degree. This operator commutes with $\mathcal{L}_\mathcal{T}$.

Let $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ be the kernel of $\Box_{b,q}$ in $L^2(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*)$,

$$\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) = \{\phi \in L^2(\mathcal{N};\textstyle\bigwedge^q\overline{\mathcal{K}}^*) : \Box_{b,q}\phi = 0\}.$$

In each degree, the operator $-i\mathcal{L}_\mathcal{T}$, viewed initially as acting on distributional sections, gives by restriction an operator on $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ with values in distributional sections in the kernel of $\Box_{b,q}$. Let

$$\operatorname{Dom}(\mathcal{L}_\mathcal{T}) = \{\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) : -i\mathcal{L}_\mathcal{T}\phi \in L^2(\mathcal{N};\textstyle\bigwedge^q\overline{\mathcal{V}}^*)\}$$

Thus $\mathcal{L}_\mathcal{T}\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ if $\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$

Proposition 7.4. *The operator*

$$-i\mathcal{L}_\mathcal{T} : \operatorname{Dom}(\mathcal{L}_\mathcal{T}) \subset \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \to \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}), \tag{7.5}$$

is Fredholm selfadjoint with compact resolvent. Hence $\operatorname{spec}(-i\mathcal{L}_\mathcal{T})$ *is a closed discrete subset of* $\mathbb{R}$ *and there is an orthogonal decomposition*

$$\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) = \bigoplus_{\tau \in \operatorname{spec}(-i\mathcal{L}_\mathcal{T})} \mathscr{H}^q_{\overline{\partial}_b,\tau}(\mathcal{N})$$

where

$$\mathscr{H}^q_{\overline{\partial}_b,\tau}(\mathcal{N}) = \{\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) : -i\mathcal{L}_\mathcal{T}\phi = \tau\phi\}.$$

It is immediate that (7.5) is densely defined.

Proof. The operator $\Box_{b,q} - \mathcal{L}^2_\mathcal{T}$ is a nonnegative symmetric operator when viewed in the space of smooth sections. Furthermore, it is elliptic. To see this let $\iota : \overline{\mathcal{K}}_\beta \to \mathbb{C}T\mathcal{N}$ be the inclusion map. The kernel of dual map $\iota^* : \mathbb{C}T^*\mathcal{N} \to \overline{\mathcal{V}}^*$ intersects $T^*\mathcal{N}$ (the real covectors) in exactly the characteristic variety of $\overline{\mathcal{K}}_\beta$, the span of the form θ_β. The principal symbol of $\overline{\partial}_b$ at $\boldsymbol{\xi} \in T^*\mathcal{N}$ is $\boldsymbol{\sigma}(\overline{\partial}_b)(\boldsymbol{\xi})(\phi) = i(\iota^*\boldsymbol{\xi}) \wedge \phi$, so just as for the standard Laplacian, $\boldsymbol{\sigma}(\Box_{b,q})(\boldsymbol{\xi}) = \|\iota^*(\boldsymbol{\xi})\|^2 I$ where the norm is the one induced on $\overline{\mathcal{V}}^*$ by that of $\overline{\mathcal{V}}$. So $\boldsymbol{\sigma}(\overline{\partial}_b)(\boldsymbol{\xi})$ is nonnegative, and vanishes to exactly order 2 on $\operatorname{Char}\overline{\mathcal{K}}_\beta$. The principal symbol of $-\mathcal{L}_\mathcal{T}^2$ is $\boldsymbol{\sigma}(-\mathcal{L}_\mathcal{T}^2)(\boldsymbol{\xi}) = (\langle \boldsymbol{\xi}, \mathcal{T}\rangle)^2 I$, hence $\boldsymbol{\sigma}(-\mathcal{L}_\mathcal{T}^2)(\boldsymbol{\xi})$ is positive when $\boldsymbol{\xi}$ is nonzero and proportional to θ. Thus

$$\boldsymbol{\sigma}(\square_{b,q} - \mathcal{L}_\mathcal{T}^2)(\boldsymbol{\xi})$$

is invertible for any $\boldsymbol{\xi} \in T^*\mathcal{N}\backslash 0$. This analysis also leads to the conclusion that $\operatorname{Dom}(\mathcal{L}_\mathcal{T})$ is a subspace of the Sobolev space $H^1(\mathcal{N}; \bigwedge^q \overline{\mathcal{V}}^*)$.

Using ellipticity and that $\square_{b,q} - \mathcal{L}_\mathcal{T}^2$ is symmetric we deduce the existence of a parametrix B so that

$$B(\square_{b,q} - \mathcal{L}_\mathcal{T}^2) = (\square_{b,q} - \mathcal{L}_\mathcal{T}^2)B = I - \Pi_q$$

where Π is the orthogonal projection on $H = \ker(\square_{b,q} - \mathcal{L}_\mathcal{T}^2)$, a finite dimensional space consisting of smooth sections. The operator

$$B : L^2(\mathcal{N}; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to L^2(\mathcal{N}; \textstyle\bigwedge^q \overline{\mathcal{V}}^*)$$

is pseudodifferential of order -2, selfadjoint, and commutes with $\mathcal{L}_\mathcal{T}$, hence with $\square_{b,q}$. In particular, it maps $\ker \square_{b,q}$ into $\ker \square_{b,q}$, that is, $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ into itself. If $\phi \in H$, then

$$\|\overline{\partial}_b \phi\|^2 + \|\overline{\partial}_b^\star \phi\|^2 + \|\mathcal{L}_\mathcal{T}\phi\|^2 = 0,$$

so $\phi \in \mathscr{H}^q_{\overline{\partial}}(\mathcal{N})$ and $\mathcal{L}_\mathcal{T}\phi = 0$. In particular, $H \subset \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ and we may view the restriction of Π to $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ as a finite rank projection $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \to \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ (mapping into $\operatorname{Dom}(\mathcal{L}_\mathcal{T})$). Suppose $\phi \in \operatorname{Dom}(\mathcal{L}_\mathcal{T})$. Then

$$[B(-i\mathcal{L}_\mathcal{T})](-i\mathcal{L}_\mathcal{T})\phi = -B\mathcal{L}_\mathcal{T}^2\phi = B(\square_{b,q} - \mathcal{L}_\mathcal{T}^2)\phi = \phi - \Pi\phi \tag{7.6}$$

using that $\operatorname{Dom}(\mathcal{L}_\mathcal{T}) \subset \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$. Since B commutes with $\mathcal{L}_\mathcal{T}$, we may write the equality of the left and rightmost terms also as

$$[-i\mathcal{L}_\mathcal{T}B](-i\mathcal{L}_\mathcal{T})\phi = \phi - \Pi\phi, \quad \phi \in \operatorname{Dom}(\mathcal{L}_\mathcal{T}). \tag{7.7}$$

If $\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$, then $B\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \cap H^2(\mathcal{N}; \bigwedge^q \overline{\mathcal{V}}^*)$, so

$$B\mathcal{L}_\mathcal{T}\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \cap H^1(\mathcal{N}; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \subset \operatorname{Dom}(\mathcal{L}_\mathcal{T})$$

Thus if

$$\pi : L^2(\mathcal{N}; \textstyle\bigwedge^q \overline{\mathcal{V}}) \to L^2(\mathcal{N}; \textstyle\bigwedge^q \overline{\mathcal{V}}), \quad \iota : \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \to L^2(\mathcal{N}; E)$$

are, respectively, the orthogonal projection on $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ and the inclusion map, then (7.6), (7.7) give that $S = -i\pi\mathcal{L}_\mathcal{T}B\iota$ is a parametrix for (7.5), compact because $\mathcal{L}_\mathcal{T}B$ is of order -1.

We now show that $\operatorname{Dom}(\mathcal{L}_\mathcal{T})$ is dense in $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$. Let $\psi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ be orthogonal to $\operatorname{Dom}(\mathcal{L}_\mathcal{T})$. If $\phi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ then $B\mathcal{L}_\mathcal{T}\phi \in \operatorname{Dom}(\mathcal{L}_\mathcal{T})$, so $(B\mathcal{L}_\mathcal{T}\phi, \psi) = 0$. Since

$$(B\mathcal{L}_\mathcal{T}\phi, \psi) = (\phi, \mathcal{L}_\mathcal{T}B\psi)$$

and ϕ is arbitrary, we conclude that $\mathcal{L}_\mathcal{T}B\psi = 0$. Thus also $\mathcal{L}_\mathcal{T}^2 B\psi = 0$, hence $(\square_{b,q} - \mathcal{L}_\mathcal{T}^2)B\psi = 0$. Consequently $\psi = \Pi\psi$, hence $\psi \in \operatorname{Dom}(\mathcal{L}_\mathcal{T})$. Therefore $\psi = 0$. Thus (7.5) is a densely defined operator.

Finally, to prove selfadjointness of (7.5) we only need to verify that its deficiency indices vanish. This can be accomplished as follows. Suppose

$$B(\lambda)(\square_{b,q} - \mathcal{L}_\mathcal{T}^2 - \lambda^2) = I.$$

This formula can be viewed as holding in the Sobolev space $H^1(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*)$, and gives

$$B(\lambda)(-\mathcal{L}_{\mathcal{T}}^2-\lambda^2)\phi=\phi,\quad \phi\in\mathrm{Dom}(\mathcal{L}_{\mathcal{T}})$$

since $\mathrm{Dom}(\mathcal{L}_{\mathcal{T}})\subset H^1(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*)$. Writing this as

$$[B(\lambda)(-i\mathcal{L}_{\mathcal{T}}+\lambda)](-i\mathcal{L}_{\mathcal{T}}-\lambda)\phi=\phi,\quad \phi\in\mathrm{Dom}(\mathcal{L}_{\mathcal{T}})$$

and using that $[B(\lambda)(-i\mathcal{L}_{\mathcal{T}}+\lambda)]$ commutes with $(-i\mathcal{L}_{\mathcal{T}}^2-\lambda)$ we see that the resolvent set of (7.5) contains $\mathbb{C}\backslash\mathbb{R}$.

This completes the proof of the proposition. □

The proof of Theorem 7.1 will also require a rough Weyl estimate. The main ingredient is:

Lemma 7.8. *Let $\{\phi_j\}_{j\in J}$ be an orthonormal basis of $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ consisting of eigenvectors of $-i\mathcal{L}_{\mathcal{T}}$, $\phi_j\in\mathscr{H}^q_{\overline{\partial}_b,\tau_j}(\mathcal{N})$. Then there are positive constants C and μ such that*

$$\|\phi_j(p)\|\le C(1+|\tau_j|)^\mu \quad \text{for all } p\in\mathcal{N},\ j\in J. \tag{7.9}$$

If $\psi\in C^\infty(\mathcal{N};\bigwedge^q\overline{\mathcal{V}}^*)$, then for each positive integer N there is C_N (depending on ψ) such that

$$(\psi,\phi_j)\le C_N(1+|\tau_j|)^{-N} \quad \text{for all } j. \tag{7.10}$$

Proof. The proof is similar to that of the analogous statement for elliptic selfadjoint operators. The ellipticity of $\Box_{b,q}-\mathcal{L}_{\mathcal{T}}^2$ gives the a priori estimates

$$\|\phi\|^2_{s+m}\le C_{s+m}(\|\Box_{b,q}\phi-\mathcal{L}_{\mathcal{T}}^2\phi\|_s^2+\|\phi\|_s^2),\quad \phi\in H^{s+m}(\mathcal{N};E)$$

for any s. Replacing ϕ_j for ϕ gives

$$\|\phi_j\|^2_{s+2}\le C_{s+2}(\|\tau_j^2\phi_j\|_s^2+\|\phi_j\|_s^2),$$

that is,

$$\|\phi_j\|^2_{s+2}\le C_{s+2}(1+|\tau_j|^4)\|\phi_j\|_s^2.$$

By induction, there is, for each $k\in\mathbb{N}$, a constant C'_k such that

$$\|\phi_j\|^2_{2k}\le C'_k(1+|\tau_j|^4)^k\|\phi_j\|^2_0$$

With k large enough the Sobolev embedding theorem gives

$$\|\phi_j\|^2_{L^\infty}\le C(1+|\tau_j|^4)^k\|\phi_j\|_0^2 \quad \text{for all } j\in J \tag{7.11}$$

with some constant C. This proves (7.9), since $\|\phi_j\|_0=1$. To prove the second statement let $\psi\in C^\infty(\mathcal{N};E)$ and pick an integer N. Then

$$|\tau_j|^N|(\phi_j,\psi)|=|(\mathcal{L}_{\mathcal{T}}^N\phi_j,\psi)|=|(\phi_j,\mathcal{L}_{\mathcal{T}}^N\psi)|\le\|\phi_j\|_0\,\|\mathcal{L}_{\mathcal{T}}^N\psi\|.$$

Then (7.10) follows, since $\|\phi_j\|_0=1$. □

The estimates (7.11) can be used as in an argument of W. Allard presented in Gilkey [5, Lemma 1.6.3], see also [10, Proposition 1.4.7], to prove:

Lemma 7.12. *There are positive constants C and μ such that*

$$\dim\bigoplus_{\substack{\tau_0\in\mathrm{spec}_0(-i\mathcal{L}_{\mathcal{T}})\\|\tau_0|<\tau}}\mathcal{E}_{\tau_0}\le C\tau^\mu.$$

This and the estimates (7.10) give:

Lemma 7.13. *Let $\{\phi_j\}_{j\in J}$ be an orthonormal basis of $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ consisting of eigenvectors of $-i\mathcal{L}_{\mathcal{T}}$. If $\psi \in \mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N}; E)$, then the Fourier series*

$$\psi = \sum_{j\in J} (\psi, \phi_j)\phi_j$$

converges in $C^\infty(\mathcal{N}; E)$.

Of course these lemmas are of interest only when $\mathscr{H}^q_{\overline{\partial}_b}(\mathcal{N})$ is infinite dimensional.

Remark 7.14. Suppose $\mathcal{N}$ with the CR structure $\mathcal{K}_\beta$ is nondegenerate. Let $\{\phi_\ell\}_{\ell=0}^\infty$ is an orthonormal basis of $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$ consisting of eigenfunctions of the operator (7.5). Using an invariant positive density to trivialize the bundle of densities we identify generalized functions and densities. If u is a CR distribution, then

$$u = \sum \langle u, \overline{\phi}_\ell\rangle \phi_\ell$$

with convergence in the space of generalized functions. This may be interpreted as a global version of the Baouendi-Treves Approximation formula [1] when written as

$$u = \lim_{L\to\infty} \sum_{\ell=0}^{L} \langle u, \overline{\phi}_\ell\rangle \phi_\ell.$$

Proof of Theorem 7.1. Since Levi_θ is definite, the space $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N})$ is infinite dimensional. Let $\{\phi_\ell\}_{\ell=0}^\infty$ be an orthonormal basis of $\mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$ as in Example 7.14. Then properties (a-b) on page 16 imply

(1) for all $p_0 \in \mathcal{N}$, $\operatorname{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \dots\}$ is the annihilator of $\overline{\mathcal{K}}_\beta$ in $\mathbb{C}T^*_{p_0}\mathcal{N}$;
(2) the functions ϕ_ℓ, $\ell = 1, 2, \dots$ separate points of $\mathcal{N}$.

This is proved in the same way as the analogous two statements in the proof of Theorem 1.1, taking advantage of the fact that if $f \in \mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N}) \cap C^\infty(\mathcal{N})$, then the Fourier series

$$f = \sum_\ell (f, \phi_\ell)\phi_\ell$$

converges in $C^\infty(\mathcal{N})$, see Lemma 7.13. As in the proof of Theorem 1.1, we conclude that there is an embedding

$$F : \mathcal{N} \to \mathbb{C}^N$$

whose components ζ^j are CR functions with respect to $\overline{\mathcal{K}}_\beta$ and satisfy $-i\mathcal{T}\zeta^j = \tau_j\zeta^j$. We assume, making full use of (2), that the differentials of these component functions span the annihilator of $\mathcal{K}_\beta$ at each $p \in \mathcal{N}$. By Lemma 5.1,

$$\overline{\mathbb{D}}\zeta^j + i\sigma_j\zeta^j\beta = 0, \quad j = 1, \dots, N$$

with $\sigma_j = -i\tau_j$. The map $\tilde{F}$ in constructed from F as in Proposition 6.1 then has components which are CR with respect to $\beta' = \beta + \overline{\mathbb{D}}u$ and maps into S^{2N-1}. By Proposition 6.9, $\tilde{F}$ is an immersion. However, while F is injective, $\tilde{F}$ may not be. We will correct this by increasing the number of components of F.

Let $w^1, \dots, w^N$ be the complex coordinates in $\mathbb{C}^N$. The vector fields

$$R = \sum_j \tau_j(w^j\partial_{w^j} + \overline{w}^j\partial_{\overline{w}^j}), \quad \mathcal{T}' = i\sum_j \tau_j(w^j\partial_{w^j} - \overline{w}^j\partial_{\overline{w}^j})$$

on $\mathbb{C}^N$ are real and commute, so give a foliation $\mathcal{F}$ of $\mathbb{C}^N\backslash 0$ by real 2-dimensional submanifolds. Since $JR = T$, the leaves are 1-dimensional complex (immersed) submanifolds of $\mathbb{C}^N\backslash 0$. The leaves are parametrized by their intersection with S^{2N-1}, each intersection being an orbit of $\mathcal{T}'$ in the sphere (the leaves are analogues of the complex lines forming $\mathbb{CP}^{N-1}$). For $\varrho \in \mathbb{C}$ and $w \in \mathbb{C}^N\backslash 0$ define

$$\varrho \cdot w = (e^{\tau_1 \varrho} w^1, \dots, e^{\tau_N \varrho} w^N).$$

For each $\varrho \in \mathbb{C}$ and $w \in \mathbb{C}^N\backslash 0$, $\varrho \cdot w$ belongs to the leaf passing through w. Since $\mathcal{T}\zeta^j = i\tau_j \zeta^j$, $F_*\mathcal{T} = \mathcal{T}'$, so F maps orbits to orbits. In particular F maps orbits of $\mathcal{T}$ into leaves of the foliation. Since the components of $\tilde{F}$ are $e^{-i\sigma_j u}\zeta^j = e^{-\tau_j u}\zeta^j$,

$$\tilde{F}(p) = -u(p) \cdot F(p),$$

which means that $\tilde{F}(p)$ lies in the intersection of the leaf containing $F(p)$ and the unit sphere. Using that F is injective it is easy to see that the restriction of $\tilde{F}$ to any orbit of $\mathcal{T}$ is injective. But it may happen that points p_0, $p_1 \in \mathcal{N}$ on different orbits of $\mathcal{T}$ are mapped by F to the same leaf of $\mathcal{F}$, so the two orbits are mapped to the same orbit by $\tilde{F}$ with the effect that $\tilde{F}$ is not injective. To solve this problem we will increase the number of components of the original map F.

Let $Z = \{(p_0, p_1) \in \mathcal{N} \times \mathcal{N} : p_0 \neq p_1,\ \tilde{F}(p_0) = \tilde{F}(p_1)\}$. We show that this is a closed set. Suppose $\{(p_{0,k}, p_{1,k})\}$ is a sequence in Z that converges in $\mathcal{N} \times \mathcal{N}$ to some point (p_0, p_1). By continuity, $\tilde{F}(p_0) = \tilde{F}(p_1)$. We will show that $p_0 \neq p_1$, and thus we conclude $(p_0, p_1) \in Z$. Suppose, to the contrary, that $p_0 = p_1$. Since $\tilde{F}$ is an immersion, p_0 has a neighborhood U with the property that $(p'_0, p'_1) \in U \times U$ and $\tilde{F}(p'_0) = \tilde{F}(p'_1)$ imply $p'_0 = p'_1$. This contradicts the existence of a sequence in Z converging to (p_0, p_0). Thus no point on the diagonal in $\mathcal{N} \times \mathcal{N}$ belongs to Z, hence Z is indeed closed.

More generally, Z contains no pair (p_0, p_1) such that $p_1 \in \mathcal{O}_{p_0}$, the orbit of $\mathcal{T}$ through p_0. For if the latter relation holds for $(p_0, p_1) \in Z$, then $\tilde{F}(p_0) = \tilde{F}(p_1)$ gives $u(p_0) \cdot F(p_0) = u(p_1) \cdot F(p_1)$, but since $u(p_0) = u(p_1)$ (because $\mathcal{T}u = 0$), $F(p_0) = F(p_1)$. Since F is injective, $p_0 = p_1$, but we have already concluded that W contains no point of the diagonal of $\mathcal{N} \times \mathcal{N}$.

If $(p_0, p_1) \in Z$, then $\tilde{F}(p_0) = \tilde{F}(p_1)$, so $F(p_0)$ and $F(p_1)$ belong to the same leaf of $\mathcal{F}$: Therefore there is $\varrho \in \mathbb{C}$ such that $F(p_0) = \varrho \cdot F(p_1)$, that is,

$$\zeta^j(p_0) = e^{\tau_j \varrho} \zeta^j(p_1), \quad j = 1, \dots, N. \tag{7.15}$$

If the real part of ϱ vanishes, then $F(p_1)$ and $F(p_0)$ belong to the same orbit of $\mathcal{T}'$, so p_0 and p_1 belong to the same orbit of $\mathcal{T}$ since F is injective. But then $p_0 = p_1$, contradicting $(p_0, p_1) \in Z$. So $\Re\varrho \neq 0$. We will show later that

(3) if $\Re\varrho \neq 0$ then $\{(p_0, p_1) : \phi_\ell(p_0) = e^{\tau_\ell \varrho}\phi_\ell(p_1) \ \text{ for all } \ell\}$ is empty.

Granted this, we proceed as follows. Pick $(p_0, p_1) \in Z$. Associated with this pair there is a number $\varrho(p_0, p_1)$ with $\Re\varrho(p_0, p_1) \neq 0$ such that (7.15) holds. Pick ℓ such that

$$\phi_\ell(p_0) \neq e^{\tau_\ell \varrho(p_0, p_1)} \phi_\ell(p_1) \tag{7.16}$$

taking advantage of (3). Fix some j_0 such that $\zeta^{j_0}(p_1) \neq 0$. Such j_0 exists because of Part (2) of Proposition 6.9 There is a neighborhood U of (p_0, p_1) in $\mathcal{N} \times \mathcal{N}$ in which there is a unique continuous function $\varrho : U \to \mathbb{C}$ such that

$$\zeta^j(q_0) = e^{\tau_j \varrho(q_0, q_1)} \zeta^{j_0}(q_1), \quad (q_0, q_1) \in U$$

By continuity and because of (7.16), we may assume

$$\phi_\ell(q_0) \neq e^{\tau_\ell \varrho(q_0,q_1)} \phi_\ell(q_1), \quad (q_0, q_1) \in U$$

shrinking U if necessary. Then, if F is augmented with the function ϕ_ℓ, (7.15) will cease to hold for $(q_0, q_1) \in U$ and all the component functions of the augmented map. Since Z is compact, we can cover it with finitely many such open sets and augment the map F to a map $F' : \mathcal{N} \to \mathbb{C}^{N'}$ for which the construction of Proposition 6.1 gives an injective map $\tilde{F}' : \mathcal{N} \to S^{2N'-1}$, hence an embedding. Indeed, if $\tilde{F}'(p_0) = \tilde{F}'(p_1)$, then, if p_0 and p_1 lie in the same orbit of $\mathcal{T}$ then $p_0 = p_1$, and if p_0 and p_1 lie in different orbits, then (7.15) holds with $j = 1, \dots, N'$, in particular $j = 1, \dots, N$, with some ϱ with nonzero real part (determined by F, p_0 and p_1). So $(p_0, p_1) \in Z$, hence for some ζ^j with $j > N$ we must have $\zeta^j(p_0) \neq e^{\tau_j \varrho} \zeta^j(p_1)$, contradicting (7.15).

To complete the proof we show the validity of (3) (see page 21). Let $\varrho \in \mathbb{C}$ be such that $\Re\varrho \neq 0$. We will assume that there is (p_0, p_1) such that

$$\forall \ell : \ \phi_\ell(p_0) = e^{\tau_\ell \varrho} \phi_\ell(p_1) \tag{7.17}$$

and derive a contradiction. We first note that $p_0 \neq p_1$, since there is ℓ such that $\phi_\ell(p_0) \neq 0$ (and $\Re\varrho \neq 0$). If $\Re\varrho > 0$, exchange p_0 and p_1, so we may assume that (7.17) holds with $\Re\varrho < 0$. By Part (1) of Proposition 6.9, all τ_ℓ have the same sign. Changing $\mathcal{T}$ to $-\mathcal{T}$ (and β to $-\beta$ for the sake of consistency) if necessary we may assume that all τ_j are positive; this is already the case if $\mathrm{Levi}_{\theta_\beta}$ is positive definite, but we do not need this fact in our proof.

The estimate (7.9) applied to $\phi_\ell(p_1)$ gives

$$|\phi_\ell(p_0)| \leq C e^{\tau_\ell \Re\varrho/2} \tag{7.18}$$

for some $C > 0$. Suppose $u \in \mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$. Then u has a restriction to the orbit through p_0. Let $\iota : \mathbb{R} \to \mathcal{N}$ be the map $\iota(t) = \mathfrak{a}_t(p_0)$. Let $W = \mathrm{Char}\,\Box_{b,0}$. The Fourier series $u = \sum_\ell u_\ell \phi_\ell$, $u_\ell = (u, \phi_\ell)$, converges in $C^{-\infty}_W(\mathcal{N})$ because $\Box_{b,0} \sum_{\ell=0}^k u_\ell \phi_\ell = 0$ for all k and $\Box_{b,0}$ is elliptic off of W. So, since $\iota^* : C^{-\infty}_W(\mathcal{N}) \to C^{-\infty}(\mathbb{R})$ is continuous, $\iota^* u = \sum u_\ell e^{i\tau_\ell t} \phi_\ell(p_0)$ in $C^{-\infty}(\mathbb{R})$. Let $\chi \in C^\infty_c(\mathbb{R})$. The Fourier transform of $\chi\iota^* u$ is

$$\sum_\ell u_\ell \widehat{\chi}(\tau - \tau_\ell) \phi_\ell(p_0)$$

and the estimates (7.18) imply that $(\chi\iota^* u)\widehat{\ }(\tau)$ is rapidly decreasing in τ (since $\Re\varrho < 0$). Thus $\iota^* u$ is smooth.

We will now show that there is $u \in \mathscr{H}^0_{\overline{\partial}_b}(\mathcal{N})$ such that $\iota^* u$ is not smooth using a support function for the CR structure at p_0 and a well known trick used in the study of hypoelliptic operators. Let (z, t) be a hypoanalytic chart for the structure $\overline{\mathcal{V}}$ centered at p_0, mapping its domain U to $B \times I$ where B is an open ball in $\mathbb{C}^n$ centered at 0 and $I \subset \mathbb{R}$ is an open interval around 0. The vector fields $\partial_{\overline{z}^\mu}$, $\mu = 1, \dots, n$, ∂_t, form a frame of $\overline{\mathcal{V}}$ over U with dual frame $\overline{\mathbb{D}}\overline{z}^\mu$, $\overline{\mathbb{D}}t$, and

$$\beta = \sum_{\mu=1}^n \beta_\mu \overline{\mathbb{D}}\overline{z}^\mu - i\overline{\mathbb{D}}t.$$

Since $\overline{\mathbb{D}}\beta = 0$, the coefficients β_μ are independent of t. Let

$$t' = t + 2\Re\Big[i\Big(\sum_{\mu=1}^{n} \beta_\mu(p_0)\overline{z}^\mu + \frac{1}{2}\sum_{\mu,\nu=1}^{n} \frac{\partial\beta_\mu}{\partial\overline{z}^\nu}(p_0)\overline{z}^\mu\overline{z}^\nu\Big)\Big].$$

Since $\partial_{\overline{z}^\nu}\beta_\mu = \partial_{\overline{z}^\mu}\beta_\nu$ (because $\overline{\mathbb{D}}\beta = 0$),

$$i\beta - \overline{\mathbb{D}}t' = i\sum_{\mu=1}^{n}\Big(\beta_\mu - \beta_\mu(p_0) - \sum_{\nu=1}^{n}\frac{\partial\beta_\mu}{\partial\overline{z}^\nu}(p_0)\overline{z}^\nu\Big)\overline{\mathbb{D}}\overline{z}^\mu.$$

The right hand side is $\overline{\mathbb{D}}$-closed, since the left hand side is, and since the right hand side is independent of t and $\overline{\mathbb{D}}t$, the form

$$b = i\sum_{\mu=1}^{n}\Big(\beta_\mu - \beta_\mu(p_0) - \sum_{\nu=1}^{n}\frac{\partial\beta_\mu}{\partial\overline{z}^\nu}(p_0)\overline{z}^\nu\Big)d\overline{z}^\mu$$

is $\overline{\partial}$-closed. Let α solve $\overline{\partial}\alpha = b$ in B and let

$$g = \alpha + t' - \alpha(p_0) - \sum_{\mu=1}^{n}\frac{\partial\alpha}{\partial z^\mu}(p_0)z^\mu - \frac{1}{2}\sum_{\mu,\nu=1}^{n}\frac{\partial^2\alpha}{\partial z^\mu\partial z^\nu}(p_0)z^\mu z^\nu$$

Then

$$\overline{\mathbb{D}}g = i\beta,$$

so $\overline{\mathbb{D}}g$ vanishes on $\overline{\mathcal{K}}_\beta$: g is a CR function.

It is easily verified that

$$g = t' + i\sum_{\mu,\nu}\frac{\partial\beta_\mu}{\partial z_\nu}(p_0)z^\nu\overline{z}^\mu + \mathcal{O}(|z|^3).$$

On the other hand, the form θ_β is given by

$$\theta_\beta = dt + i\sum_{\mu=1}^{n}\beta_\mu d\overline{z}^\mu - i\sum_{\mu=1}^{n}\overline{\beta}_\mu dz^\mu,$$

and

$$-id\theta_\beta = \sum_{\mu,\nu=1}^{n}\Big[\frac{\partial\beta_\mu}{\partial z^\nu} - \frac{\overline{\beta}_\mu}{\partial\overline{z}^\nu}\Big]dz^\nu\wedge d\overline{z}^\mu$$

using $\partial_{\overline{z}^\nu}\beta_\mu = \partial_{\overline{z}^\mu}\beta_\nu$. The vector fields

$$L_\nu = \frac{\partial}{\partial z^\nu} + i\overline{\beta}_\nu\frac{\partial}{\partial t}, \quad \nu = 1,\dots,n$$

form a frame for $\mathcal{K}_\beta$ in U, and by hypothesis $\mathrm{Levi}_{\theta_\beta}$ is positive definite. So the matrix with coefficients

$$-id\theta_\beta(L_\nu,\overline{L}_\mu) = \frac{\partial\beta_\mu}{\partial z^\nu} - \frac{\overline{\beta}_\mu}{\partial\overline{z}^\nu}$$

is positive definite. It follows that the quadratic part of

$$\Im g = -\frac{i}{2}\sum_{\mu.\nu=1}^{n}\Big(\frac{1}{\beta_0}\frac{\partial\beta_\mu}{\partial z^\nu} - \frac{1}{\overline{\beta}_0}\frac{\overline{\beta}_\mu}{\partial\overline{z}^\nu}\Big)z^\nu\overline{z}^\mu + \mathcal{O}(|z|^3)$$

at p_0 is positive definite. Thus shrinking B we may assume that

$$\Im g \geq c|z|^2 \text{ for some } c > 0.$$

Define

$$u_0 = \int_0^\infty e^{i\tau g}(1+\tau^2)^{-1}d\tau.$$

in U. The function u_0 is CR (since g is) and in L^2_{loc}, but not in $C^\infty(U)$. In fact, $\mathrm{WF}(u_0) = \{\tau\theta_\beta(p_0) \in T^*_{p_0}\mathcal{N} : \tau > 0\}$. Let $\chi \in C_c^\infty(U)$ be equal to 1 near p_0 and let G be Green's operator for $\Box_{b,1}$. The operator G, being a pseudodifferential operator of type $(1/2, 1/2)$, preserves wavefront set. Therefore, since $\overline{\partial}_b\chi u_0$ is smooth, so is $\overline{\partial}_b^\star G\overline{\partial}_b\chi u_0$. Let

$$u = \chi u_0 - \overline{\partial}_b^\star G\overline{\partial}_b\chi u_0.$$

The pull-back of $\overline{\partial}_b^\star G\overline{\partial}_b\chi u_0$ to the orbit through p_0 is smooth. The orbit through p_0 intersects U on sets $z = $ const., in particular $\{(z,t) : z = 0\}$ is part of the orbit. On the latter set, $g = t$, therefore the pull-back of χu_0 is equal to

$$\int_0^\infty e^{i\tau t}(1+\tau^2)^{-1}d\tau$$

near $t = 0$, which is not smooth. Thus for no pair (p_0, p_1) does (7.17) hold. $\square$

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