RESEARCH OVERVIEW

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The central goal of my research is to weave together three different points of view in low-dimensional topology. These viewpoints are

- *Combinatorial topology*: This includes the study of knots as depicted by planar diagrams, and of 3–manifolds defined by combinatorial constructions such as triangulations, surgery presentations, or fibrations over the circle.
- *Hyperbolic geometry*: Following the work of Thurston and Perelman, we know that "almost every" 3–manifold admits a metric of constant curvature -1. Because these metrics are unique up to isometry, geometric quantities provide a rich source of topological invariants.
- *Quantum topology*: Another rich source of invariants, motivated by quantum physics, is the Jones polynomial of knots, as well and its descendants and generalizations.

At present, each of these viewpoints is fairly well developed, and has led to the resolution of many open problems. However, connections among these points of view are still tenuous.

One of the central open problems in low-dimensional topology is to to construct an effective, computable dictionary between the combinatorial and geometric features of a 3–manifold. It is a consequence of Perelman's geometrization theorem [62, 63] and Mostow's rigidity theorem [58, 65] that such a dictionary must exist. However, understanding exactly *how* topological and combinatorial invariants relate to geometric invariants remains a hard and elusive problem. My work with several collaborators, outlined below, provides some of the first two-sided estimates on geometric quantities in terms of combinatorial data. These estimates hold in infinite but somewhat restricted families of manifolds; part of an ongoing project is to make them more universally applicable.

A second major open problem is to understand the geometric meaning of quantum invariants such as the Jones polynomial. This problem dates to the late 1980s, when Witten interpreted the Jones polynomial in terms of quantum gravity and conjectured a close relationship between this invariant and geometric structures on 3–manifolds [74]. In a similar vein, the volume conjecture of Kashaev and Murakami–Murakami postulates a precise relationship between colored Jones polynomials and hyperbolic volume [45, 59]. While these conjectures are still wide open, my work with Kalfagianni and Purcell establishes some of the first concrete bridges among coefficients of the Jones polynomial, hyperbolic geometry, and classical geometric topology.

The sections below outline some of my results and ongoing projects. They are organized as follows:

1. Link diagrams and geometry
2. Geometric control via Dehn surgery
3. The geometry of fibered 3–manifolds, via the mapping class group
4. Relating quantum and topological invariants
5. Constructing hyperbolic metrics via angled triangulations
6. Geometric group theory

The most significant recent results, which form the basis of most of my ongoing projects, are contained in sections 3 and 4.

1. Link diagrams and geometry

The idea that combinatorics should have an explicit and effective translation into hyperbolic geometry — often summarized as "what you see is what you get" topology, or "effective geometrization" — takes a particularly clear and visual form in the study of knots in S^3. This is because knots and links are easy to describe via planar diagrams, and it is natural to look for combinatorial features of the diagrams that will predict geometric invariants, such as volume or the length of geodesics.

One diagrammatic invariant that appears closely connected to hyperbolic geometry is called the *twist number*. A *twist region* is a section of a diagram $D(K)$ in which two strands of K wrap around each other maximally. (See Figure 1, left.) The number of these regions is called the *twist number* of $D(K)$, and is denoted $\mathrm{tw}(D)$. With this definition, Lackenby showed that the the twist number of an alternating diagram is approximately equal to the hyperbolic volume of the link complement [48]. More precisely, volume is bounded above and below by explicit linear functions of $\mathrm{tw}(D)$.

In a series of joint papers with Kalfagianni and Purcell, I have extended Lackenby's result to many families of non-alternating links. These include Montesinos links [34], Conway sums of alternating tangles [30], highly twisted links (where every twist region contains at least 7 crossings) [29], and positive braids with at least 3 crossings per twist region ([34], and see Theorem 8 below). In each of these cases, the hyperbolic volume of the link complement is bounded above and below by explicit linear functions of $\mathrm{tw}(D)$.

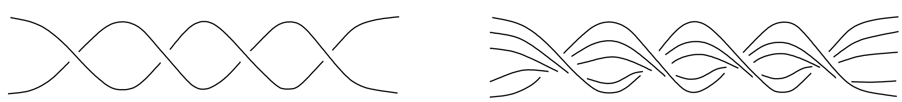

Figure 1. Left: a twist region. Right: a generalized twist region.

We also proved variants of these results that rely on a generalization of the twist number. Define a *generalized twist region* to be a region where $n \geq 2$ strands of K wrap around each other maximally (see Figure 1, right) and the *generalized twist number* $\mathrm{tw}_{\mathrm{gen}}(D)$ to be the number of these regions. Using this generalized notion, we proved two-sided diagrammatic estimates on the volume of closed 3–braids [31] and so-called double coil knots [32]. Related work appears in [13, 25].

Some of these estimates are quite sharp. For instance, here is a sample theorem.

Theorem 1 ([31]). *Let K be a hyperbolic link obtained as the closure of a 3–string braid, and let $D(K)$ be the diagram corresponding to the Schreier normal form of the braid word. Then*

$$2v_3\, \mathrm{tw}_{\mathrm{gen}}(D) - 279 \;<\; \mathrm{vol}(S^3 \smallsetminus K) \;<\; 2v_8\, \mathrm{tw}_{\mathrm{gen}}(D),$$

where $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

Furthermore, the multiplicative constants in both the upper and lower bounds are sharp.

The additive constant of 279 is presumably far from sharp. Nonetheless, it is striking to see geometry and combinatorics so closely intertwined, and satisfying to know the sharp constants that constrain their relationship. In addition, Theorem 1 can also be applied to express hyperbolic volume in terms of certain coefficients of the Jones and colored Jones polynomials (see Section 4).

The proof of Theorem 1, as well as many of its relatives [29, 30, 32], uses Dehn surgery techniques. (See Section 2 for more on these techniques.) The main idea is that every link K in braid position has a *braid axis*, namely an unknot $A \subset S^3 \smallsetminus K$, whose complement $S^3 \smallsetminus A \cong D^2 \times S^1$ is foliated by disks that intersect K a constant number of times. Drilling out the axis produces a manifold $M = S^3 \smallsetminus (K \cup A)$ that fibers over S^1 with fiber an n–punctured disk, and the geometry of $S^3 \smallsetminus K$ can be understood in terms of this fibered manifold M. This works in three steps:

1. A fibered 3–manifold M is the mapping torus of a surface homeomorphism $\varphi : S \to S$. The volume of M can be understood in terms of combinatorial data about φ. When K is a 3–braid and the fiber S is a 3–punctured disk, I worked this out in joint work with Guéritaud [41, Appendix B], with sharp constants. This is the source of sharp constants in Theorem 1.
2. The meridian of the braid axis A is a simple closed curve on a boundary torus of M. Its length (on a maximal cusp in the hyperbolic metric) can also be understood combinatorially, via my joint work with Kalfagianni and Purcell [31] and with Schleimer [37]. See Section 3, and specifically Theorem 4, for more details about how these estimates work.
3. Given estimates on the volume of M, and the meridian of the braid axis A, a theorem proved with Kalfagianni and Purcell (see Theorem 2 below) controls how much the hyperbolic volume can change when we fill in the axis A and recover $S^3 \smallsetminus K$. When the meridian is long, the geometry (in particular, the volume) of $S^3 \smallsetminus K$ closely approximates that of M.

1.1. **Ongoing project: extend this three-step program to general braids.** The above three-step procedure for estimating the volume of a link complement ought to work in much greater generality than that of 3–braids. As is well-known, every knot or link $K \subset S^3$ can be represented as a closed braid with some number of strands. Drilling out the braid axis A, as above, produces a fibered manifold M, for which steps 2 and 3 already give effective results. (See Theorems 2 and 4.) The only part that does not yet work in complete generality is step 1: estimating the geometry of M in terms of a diagram of K.

Extending step 1 to general braids requires two new ingredients. First, one needs to find effective estimates on the volume of M in terms of *some* combinatorial quantity. There is already a theorem of Brock that gives linear (but ineffective) volume bounds in terms of the combinatorics of pants on the fiber surface [9, 10]. The challenge is to find explicit constants. I am actively pursuing this with Purcell and Schleimer; see Section 3.1 for more details.

Second, one needs to relate the combinatorics of link diagrams to the graph of pants decompositions in the fiber surface S. Here, Purcell and I have observed that there *some* braid diagram $D(K)$, conjugate to the given one, has its generalized twist number $\mathrm{tw}_{\mathrm{gen}}(D)$ exactly equal to the desired quantity in the pants graph. Thus the main challenge is to find the right representative in a conjugacy class in the braid group.

If the above steps can be pushed through, they would produce a very satisfying theorem that gives combinatorial volume estimates for general knots and links.

2. Geometric control via Dehn surgery

Every closed 3–manifold can be constructed by a process called *Dehn surgery* or *Dehn filling*. Start with a manifold with boundary consisting of tori (such as the complement of a knot or link in S^3), and attach a solid torus to each boundary torus. The result depends only on the *slope* (isotopy class of simple closed curve) chosen to bound a disk in the solid torus. When the initial manifold M is hyperbolic, there is a natural way to measure the *length* of this slope. This is because each toroidal end of M is a *cusp*, homeomorphic to $T^2 \times [0, \infty)$, such that each cross-sectional torus $T^2 \times \{x\}$ is a quotient of a horosphere and carries a Euclidean metric. The maximal embedded neighborhood of this sort is called a *maximal cusp*, and the metric on its boundary is uniquely determined by the topology of M. We may measure the length of a slope in this Euclidean metric.

The length of the slope typically controls the outcome of the surgery. A foundational result of Thurston [71] is that if one starts with a hyperbolic manifold M and performs Dehn filling along a sufficiently long slope s, the resulting manifold $M(s)$ will have a hyperbolic structure very close to the original. The change in geometry was further quantified by Hodgson–Kerckhoff [42] and Brock–Bromberg [11]. My collaborators and I have both added to the suite of theorems that bound how much the geometry changes, and applied these theorems to attack some long-standing problems.

One of my results, proved jointly with Kalfagianni and Purcell, is an explicit estimate on the change in volume under the operation of Dehn filling [29].

Theorem 2 ([29]). *Let M be a cusped hyperbolic 3–manifold, and s a slope on a boundary torus of M. If the length of s on a maximal horospherical torus satisfies $\ell(s) > 2\pi$, then the filled manifold $M(s)$ has a complete hyperbolic metric satisfying*

$$\left(1 - \left(\frac{2\pi}{\ell(s)}\right)^2\right)^{3/2} \mathrm{vol}(M) \ \leq \ \mathrm{vol}(M(s)) \ < \ \mathrm{vol}(M).$$

Thus, when $\ell(s)$ is sufficiently long, the volume of the filled manifold $M(s)$ is bounded below by $\mathrm{vol}(M)$ times an explicit constant close to 1, while being bounded above by $\mathrm{vol}(M)$.

To prove theorem 2, we carefully construct a negatively curved metric on $M(s)$, with explicit control over sectional curvatures, volume, and the areas of surfaces. Then, we apply volume comparison theorems of Besson, Courtois, and Gallot [8] to relate the volume of the complete hyperbolic metric on $N(s)$ to the volume of our metric.

Since its appearance, Theorem 2 has found wide applications to both geometric and topological problems. Most notably, Gabai, Meyerhoff, and Milley needed our estimate in the endgame of their project to identify the smallest–volume closed hyperbolic 3–manifold [39, 55]. Petronio and Vesnin used our result to bound the number of tetrahedra required to triangulate a particular family of manifolds [64]. Kalfagianni, Purcell, and I used Theorem 2 give diagrammatic estimates for the volumes of several large families of knots and links [29, 30, 31, 32]; see Section 1 for details. In addition, as I describe below, the negatively curved metric constructed in the course of the proof turns out to be useful in the study of Heegaard splittings and unknotting tunnels [15, 35].

Some related results about hyperbolic Dehn filling appear in the papers [28, 36].

2.1. **Geometric control of Heegaard surfaces and unknotting tunnels.** A *Heegaard splitting* of a 3–manifold M is a decomposition into two particularly simple pieces, glued along a surface. When M is closed, each piece is a handlebody, that is, a regular neighborhood of a graph. When M has boundary, each pieces of the decomposition is a slight generalization of a handlebody, called a *compression body.* The study of 3–manifolds via Heegaard splittings dates back to the 1920s, and has shed light on both topological and geometric invariants.

A longstanding principle is that the structure of a Heegaard splitting is usually preserved during Dehn filling. In particular, it follows from the work of Moriah–Rubinstein [57] and Rieck–Sedgwick [66] that for a 3–manifold M with torus boundary, the minimal–genus Heegaard splittings of M will also give the minimal–genus Heegaard splittings of a sufficiently complicated Dehn filling. However, their work does not quantify *which* fillings are sufficiently complicated for this to hold. By contrast, a recent paper of mine with Purcell provides explicit, easily checkable hypotheses that ensure the preservation of Heegaard structures [35]. We do this via a geometric argument, in particular by bounding the area of a Heegaard surface via the negatively curved metric of Theorem 2.

In recent work [15], Cooper, Purcell, and I extend this theory to the study of unknotting tunnels. Let Y be a closed 3–manifold (frequently, S^3) and K a knot in Y. An *unknotting tunnel* for K is an arc τ from K to K, such that $Y \smallsetminus (K \cup \tau)$ is a handlebody. Unknotting tunnels are closely related to Heegaard splittings, because the boundary of a regular neighborhood of $K \cup \tau$ is a genus 2 Heegaard surface for the manifold $M = Y \smallsetminus K$.

In the early 1990s, Adams [1], Sakuma, and Weeks [68] asked several open questions about how unknotting tunnels interact with the geometry of $M = Y \smallsetminus K$. Namely: is an unknotting tunnel always isotopic to a geodesic? Can it be arbitrarily long (relative to a maximal cusp in M)? Does it appear as an edge in the geometrically canonical polyhedral decomposition of M? In the decades since, there has been some incremental progress in constructing examples of geodesic tunnels [2, 7] and very long tunnels [16]. However, the most general form of these questions remains open today. Before our work, the questions were open even for knots in S^3.

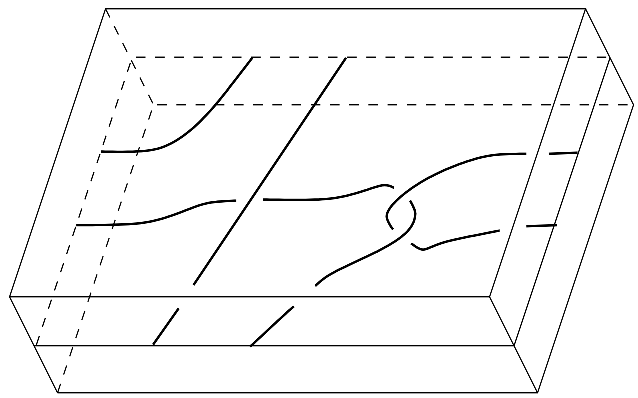

FIGURE 2. A knot K in a slab $T^2 \times [0,1]$. A sequence of embeddings of $T^2 \times [0,1]$ into S^3, all of which map T^2 to the Clifford torus, produces a sequence of hyperbolic knots K_n that have long, geodesic unknotting tunnels.

In a recent paper, Cooper, Purcell, and I provide detailed, quantitative answers to all of these questions, in the generic situation where the knot K and its complement $M = Y \smallsetminus K$ are created by sufficiently complicated Dehn filling.

Theorem 3 ([15])**.** *Let Y be a closed 3–manifold, and K a knot with hyperbolic complement $M = Y \smallsetminus K$. Suppose that M is obtained by Dehn filling a 2–cusped manifold N, where both the meridian μ and longitude λ are sufficiently long. (Here, λ is defined to be the shortest curve on a cusp torus of N that intersects μ once.)*

Then an unknotting tunnel τ for K is isotopic to a geodesic $g_\tau \subset M$. The length of this geodesic outside a maximal cusp of M is bounded above and below:

$$2 \ln \ell(\lambda) - 6 \;<\; \ell(g_\tau) \;<\; 2 \ln \ell(\lambda) + 5.$$

A satisfying feature of Theorem 3 is that it gives precise, quantitative answers to these long-standing questions. We not only construct long unknotting tunnels, but bound their length up to additive error only. Similarly, we show that τ is isotopic to the geodesic g_τ by first constructing a homotopy, and then carefully bounding the distance that points must travel during this homotopy.

Theorem 3 can be applied to construct the first known sequence of knots $K_n \subset S^3$, whose unknotting tunnels are arbitrarily long. This sequence has an explicit description, sketched in Figure 2. Each knot K_n has two unknotting tunnels, both of which are geodesic edges in the canonical triangulation of $S^3 \smallsetminus K_n$. Theorem 2, and its application to the preservation of Heegaard structures [35], are used to ensure that we have found all the unknotting tunnels of the knot K_n.

3. Geometry and combinatorics of fibered manifolds

Fibered 3–manifolds arise from a natural way to construct a manifold from lower-dimensional pieces. Given an (orientable) surface S, and an (orientation-preserving) homeomorphism $\varphi : S \to S$, the *mapping torus* of φ is

$$M_\varphi = (S \times [0,1])/(x,0) \sim (\varphi(x),1).$$

The manifold M_φ fibers over the circle with fiber S, and the map φ is called the *monodromy* of the fibration. Thurston showed that M_φ is hyperbolic if and only if $\chi(S) < 0$ and φ is *pseudo-Anosov*, which means that $\varphi^n(\gamma)$ is not homotopic to γ for any $n \neq 0$ and any essential simple closed curve $\gamma \subset S$ [70, 72]. So this construction produces a great variety of hyperbolic manifolds.

A major recent breakthrough, combining the work of Agol [3, 4], Kahn–Markovic [44], and Wise [73], is that every finite-volume hyperbolic 3–manifold has a finite cover that fibers over the circle. Because geometric properties are essentially preserved in finite covers, this result makes it possible to study the geometry of any hyperbolic 3–manifold using the structural data of a fibration.

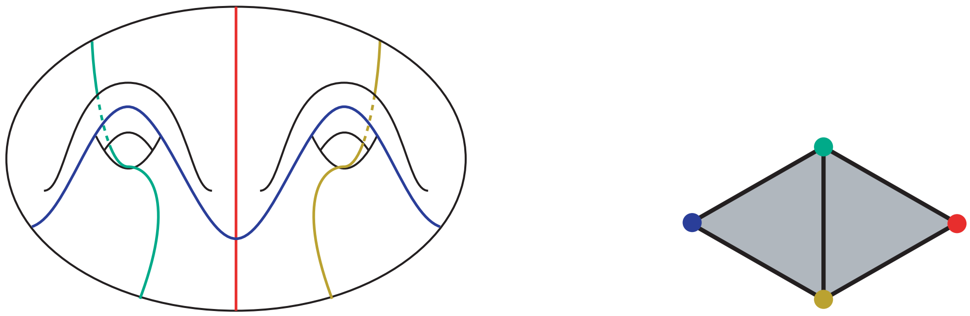

FIGURE 3. Left: essential arcs in a punctured surface S. Right: the corresponding subcomplex of the arc complex $\mathcal{A}(S)$. Disjoint arcs are connected by edges.

For those fibered 3–manifolds that are hyperbolic, the work of Minsky, Brock, and Canary on Kleinian surface groups provides a combinatorial, bi-Lipschitz model of the hyperbolic metric [56]. The bi-Lipschitz constants depend only on the fiber surface [12]. However, the existence of these constants is proved using compactness arguments; as a result the constants are unknown.

The combinatorial input for these bi-Lipschitz models comes from certain combinatorial complexes associated to S. One of these is the *curve complex* $\mathcal{C}(S)$, whose vertices are isotopy classes of essential closed curves on S, and whose edges represent disjoint curves. In a similar manner, one may define the *pants graph* $\mathcal{P}(S)$ (whose vertices are pants decompositions) and the *arc complex* $\mathcal{A}(S)$ of a surface with punctures (whose vertices are essential arcs from puncture to puncture). A small piece of the arc complex is illustrated in Figure 3.

The geometry of a fibered manifold M_φ tends to be controlled by the distance by which the map φ moves vertices in these complexes. More precisely, define the *stable translation distance* of φ to be

$$\overline{d}_{\mathcal{A}}(\varphi) = \lim_{n\to\infty} \frac{d_{\mathcal{A}}(v, \varphi^n(v))}{n}, \quad \text{for an arbitrary vertex } v \in \mathcal{A}(S),$$

and similarly for $\mathcal{C}(S)$ and $\mathcal{P}(S)$. With this definition, Brock showed that the hyperbolic volume of a mapping torus M_φ is bounded above and below by constants times the stable translation distance $\overline{d}_{\mathcal{P}}(\varphi)$ [9, 10]. The constants depend only on the fiber S. However, the constants that bound volume from below are not explicitly known, and cannot be extracted from Brock's argument.

In this context, Schleimer and I proved that when the fiber S has punctures, the translation distance of φ in the arc complex $\mathcal{A}(S)$ leads to explicit estimates on the geometry of cusps in M_φ.

Theorem 4 ([37])**.** *Let S be a surface with one puncture. Then, for any pseudo-Anosov $\varphi\colon S \to S$, the mapping torus M_φ contains a maximal cusp C, which satisfies*

$$\frac{\overline{d}_{\mathcal{A}}(\varphi)}{450\,\chi(S)^4} \;<\; \operatorname{area}(\partial C) \;\leq\; 9\,\chi(S)^2\,\overline{d}_{\mathcal{A}}(\varphi).$$

If s is the shortest closed curve on ∂C that is transverse to the fibers, then

$$\frac{\overline{d}_{\mathcal{A}}(\varphi)}{536\,\chi(S)^4} \;<\; \ell(s) \;<\; -3\,\chi(S)\,\overline{d}_{\mathcal{A}}(\varphi).$$

There is a closely analogous estimate for surfaces with multiple punctures, in terms of the complex $\mathcal{A}(S, p)$ of arcs into a particular puncture.

The family of cusped, fibered 3–manifolds is by far the most general class for which there currently exist explicit two-sided bounds on geometry in terms of combinatorics. Furthermore, the reach of Theorem 4 is extended by virtual fibration results. Given Wise's theorem that every cusped, finite-volume 3–manifold N has a fibered finite cover [73], one may use the combinatorics of that cover to gain insight into the cusp geometry of N.

3.1. **Ongoing project: find explicit volume estimates.** The proof of Theorem 4 relies on a detailed study of pleated surfaces in the fibered manifold M_φ. We sweep out the manifold M_φ by geometrically controlled surfaces, keeping track of the shortest arc in the surface as we go. This produces at least as many short arcs as the translation distance $\overline{d}_{\mathcal{A}}(\varphi)$, and each of these arcs makes a contribution to cusp area.

In a joint project with Purcell and Schleimer, we are working to extend these ideas to estimate other geometric features, including the volume of M_φ. By keeping track of *systems* of arcs rather than individual arcs, we obtain a sequence of pants decompositions, each of which makes a contribution to the volume of M_φ. When the manifold M_φ is *thick* — that is, when there is a universal lower bound on the length of the shortest geodesic — our argument gives an effective lower bound on volume that is proportional to $\overline{d}_{\mathcal{P}}(\varphi)$. This is an effective version of Brock's theorem [10].

When M_φ is not thick away from the cusp — when there are very short geodesics — our methods need to be modified somewhat. Here, our strategy is to drill out the short geodesics, and cut M_φ along surfaces containing those geodesics. This breaks the volume estimate for M_φ into estimates on several simpler, *geometrically finite* pieces. We are currently working to assemble these estimates to get effective control on the volume of the whole manifold.

3.2. **Future project: construct explicit bi-Lipschitz models.** A further, more ambitious extension of these methods would be to find explicit bi-Lipschitz constants for Minsky's combinatorial model of M_φ. One of the central difficulties in accomplishing this goal would be estimate the lengths of closed geodesics in multiple copies of the fiber surface S. Here, the thick/thin dichotomy works much as in the previous project. When M_φ is thick, our sweepout argument produces a hierarchical system of geodesics with length bounded above and below. When M_φ is thin, we plan to understand the lengths of very short geodesics via Dehn filling arguments.

4. Quantum invariants and geometric topology

In the mid-1980s, Jones invigorated low-dimensional topology by introducing the celebrated Jones polynomial [43]. This invariant of knots and links was originally constructed in terms of operator algebras, and later reformulated using the combinatorics of link projections. Since its introduction, the Jones polynomial has led to the resolution of several geometric conjectures, for instance that the crossing number of an alternating knot is realized by an alternating diagram [46, 60]. However, it has long been an open and tantalizing question to decode what exactly the Jones polynomial and its relatives say about the geometry of the link complement.

This question has produced fairly extensive conjectures. Based on physical intuition, Witten has conjectured that the Jones polynomial can be interpreted in terms of geometric structures, in particular hyperbolic structures [74]. More recently, in the late 1990s, Kashaev and Murakami–Murakami conjectured that the volume of a hyperbolic link complement can be computed as an asymptotic limit of colored Jones polynomials [45, 59]. This conjecture is still wide open.

Recall that for each link $K \subset S^3$, and for each positive integer n, the colored Jones function is a Laurent polynomial invariant of the form

$$J^n_K(t) = \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \ldots + \beta'_n t^{r_n+1} + \alpha'_n t^{r_n}. \tag{1}$$

Here, $J^2_K(t)$ is the classical Jones polynomial.

My recent work with Kalfagianni and Purcell has found that under mild diagrammatic hypotheses, both the degrees and the outer coefficients of these polynomials have clear connections to the geometric topology of $S^3 \smallsetminus K$. Theorem 6 says that the growth rate of the degree of $J^n_K(t)$, as $n \to \infty$, is actually a boundary slope of an essential surface for K. Theorem 7 says that the next-to-last coefficient β'_n is precisely the obstruction to this surface being a fiber in $S^3 \smallsetminus K$. Furthermore, the magnitude $|\beta'_n|$ provides a quantitative measurement of *how far* this surface is from being a fiber. This leads to tight, two-sided estimates on volume: for certain families of links, the coefficient β'_n determines the volume of $S^3 \smallsetminus K$ up to a factor of 4.15. This result, stated as Theorem 8, provides certain coarse evidence for the volume conjecture of Kashaev and Murakami–Murakami.

The glue that holds these theorems together is a combinatorial object called the *state graph* associated to a link diagram and its Kauffman states.

4.1. **State graphs and polynomials.** At each crossing of a diagram $D(K)$, there are two possible smoothings, or *resolutions* of the crossing, as depicted in Figure 4. A *Kauffman state* is a choice of A– or B–resolution at each crossing. Kauffman's insight [46] was that the Jones polynomial can be expressed as a sum of terms corresponding to all the states of $D(K)$.

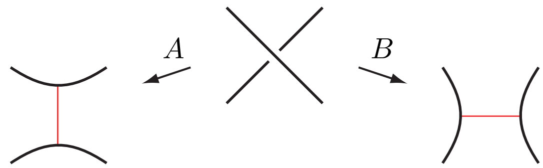

FIGURE 4. A– and B–resolutions at a crossing of D.

The choices that lead to a Kauffman state σ can be conveniently encoded in a *state graph* $\mathbb{G}_\sigma$. A state σ gives rise to a crossing–free collection of circles in the projection plane; each such circle gives rise to a vertex of $\mathbb{G}_\sigma$. Each crossing x of D gives rise to an edge between the state circles at the resolution of x. (In Figures 4 and 5, these edges are shown in red, lighter than the link projection.) Furthermore, by keeping track of an orientation on each state circle, one can obtain a cyclic ordering on the half-edges that run into each vertex of $\mathbb{G}_\sigma$. This extra structure turns $\mathbb{G}_\sigma$ into a *ribbon graph* (a graph with a 2–dimensional thickening), also called a *dessin d'enfant*.

One of my results, proved jointly with Dasbach, Kalfagianni, Lin, and Stoltzfus, is that the ribbon graph $\mathbb{G}_A$ corresponding to the all–A state carries more information than the Jones polynomial.

Theorem 5 ([19])**.** *Let $D(K)$ be a connected link diagram, and $\mathbb{G}_A$ the ribbon graph of the all–A state of D. Then a three-variable graph theoretic polynomial of $\mathbb{G}_A$, called the Bollobás–Riordan–Tutte polynomial, has a one-variable specialization that recovers the Jones polynomial $J_K(t)$.*

The essence of Theorem 5 is that the ribbon graph $\mathbb{G}_A$ encodes both invariant combinatorial data (via the Jones polynomial) and topological data. The topological connection is that the cyclic ordering around each vertex of $\mathbb{G}_A$ leads to the construction of an orientable surface, called the *Turaev surface*, onto which K has an alternating projection. This leads to a new invariant, called the *Turaev genus*, which vanishes if and only if K is alternating.

Since its appearance, the paper [19] has proved to be fairly influential, cited at least 60 times. Several authors have continued exploring the connection between ribbon graphs and algebraic knot invariants (see e.g. [14, 50]), with applications to computation [20]. There has also been a lot of ongoing interest in the Turaev genus of K. In particular, we now know [51] that the Turaev genus provides bounds on the *width* of the strip of non-zero coefficients in the knot Floer homology $\widehat{HFK}(K)$, and similarly the width of the Khovanov homology of K.

There are certain classes of diagrams for which state graphs contain even more information. Note that in a state σ, every crossing of D belongs to a region bounded by state circles. The state σ is called *homogeneous* if all crossings in the same region carry the same (A or B) resolution. The state σ is called *adequate* if every edge of $\mathbb{G}_\sigma$ connects distinct vertices. For example, the state shown in Figure 5 is both homogeneous and adequate. Finally, the diagram D is called A–adequate if the (necessarily homogeneous) all–A state is adequate.

The class of A–adequate knots is quite large: for instance, 99% of knots up to 15 crossings admit such a diagram (after possibly taking a mirror image).

For A–adequate diagrams, Theorem 5 makes it easy to read off both the degree and certain outer coefficients of the colored Jones polynomial $J^n_K(t)$. For instance, the last coefficient always satisfies

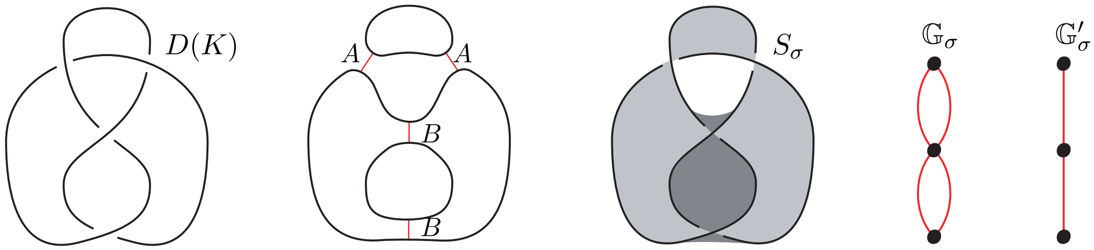

FIGURE 5. Left to right: a diagram $D(K)$. An adequate, homogeneous state σ. The state surface S_σ corresponding to σ. The graph $\mathbb{G}_\sigma$ embeds into S_σ as a spine. The reduced graph $\mathbb{G}'_\sigma$.

$|\alpha'_n| = 1$, while the next-to-last coefficient satisfies $|\beta'_n| = 1 - \chi(\mathbb{G}'_A)$. Here, $\mathbb{G}'_A$ is the *reduced graph* obtained by removing duplicate edges in $\mathbb{G}_A$. These equalities are independent of n [21, 69].

4.2. **The topology of state surfaces.** One consequence of these ideas, developed jointly with Kalfagianni and Purcell, is a connection between Jones polynomials and spanning surfaces for K.

Theorem 6 ([33]). *Let $K \subset S^3$ be a knot with an A–adequate diagram. Then there is an essential surface $S_A \subset S^3 \smallsetminus K$, whose boundary curve ∂S_A winds once around the longitude and*

$$b(S_A) \;=\; \lim_{n\to\infty} \frac{4r_n}{n^2}$$

times around the meridian. Here, r_n is the lowest degree of $J^n_K(t)$, as in equation (1)*. In words, the degrees of colored Jones polynomials determine a boundary slope of K.*

Theorem 6 establishes a large special case of a conjecture of Garoufalidis [40]: for any knot K, any limit of a subsequence of $\{4r_n/n^2\}$ should be a boundary slope of K. See Dunfield and Garoufalidis [23, 40] for other known cases of the conjecture.

Prior to this work, the main method of detecting boundary slopes relied on the character varieties of Culler and Shalen [18], which are closely connected to hyperbolic geometry. Thus it is exciting to see the same sort of geometric application arising out of quantum topology.

The surface S_A that appears in Theorem 6 is a *state surface* constructed from the all–A state of a diagram. That is: every state circle bounds a disk, and these disks can be joined by half-twisted bands to form a surface. This construction works for any state σ; see Figure 5. When σ is an adequate and homogeneous state, we know that S_σ is an essential surface [34, 61].

Because the graph $\mathbb{G}_\sigma$ embeds into S_σ as a spine, both $\mathbb{G}_\sigma$ and the reduced graph $\mathbb{G}'_\sigma$ turn out to be closely connected to the topology of the surface complement $S^3 \smallsetminus S_\sigma$.

Theorem 7 ([24, 34]). *For an A–adequate link K, the following are equivalent:*

(a) The next-to-last coefficient of the Jones polynomial of K is $\beta' = 0$.
(b) For every *A–adequate diagram $D(K)$, the reduced state graph $\mathbb{G}'_A$ is a tree.*
(c) For every *A–adequate diagram $D(K)$, $S^3 \smallsetminus K$ fibers over S^1 with fiber the corresponding state surface $S_A = S_A(D)$.*
(d) For some *A–adequate diagram $D(K)$, $S^3 \smallsetminus S_A$ is an I–bundle over $S_A(D)$.*

Taken together, Theorems 6 and 7 offer striking evidence that the degrees and coefficients of colored Jones polynomials contain quite a lot of geometric information. They lay some of the first concrete bridges between quantum invariants and geometric topology.

In fact, there is a stronger version of Theorem 7 saying that when the coefficient β' is far from 0, the surface S_A is correspondingly far from being a fiber. For any essential surface $S \subset M^3$, one may cut the surface complement $M \smallsetminus S$ along annuli into two kinds of pieces: I–bundles and complicated pieces called *guts*. This is exactly the annulus version of the JSJ decomposition. When

S is a fiber, $M \setminus S$ is an I–bundle, and the guts are empty. Otherwise, the Euler characteristic $\chi(\mathrm{guts}(M \setminus S))$ measures the "distance" between S and a fiber.

For many families of A–adequate diagrams, as well as other diagrams with an adequate and homogeneous state σ, we proved that the Euler characteristic $\chi(\mathrm{guts}(S^3 \setminus S_\sigma))$ is equal to $\chi(\mathbb{G}'_\sigma)$, the Euler characteristic of the reduced graph. This result meshes nicely with a theorem of Agol, Storm, and Thurston [6] that says the hyperbolic volume of $S^3 \setminus K$ is bounded below by a constant times $\chi(\mathrm{guts}(S^3 \setminus S_\sigma))$. As a consequence, we obtain volume estimates in terms of state graphs, hence in terms of Jones polynomial coefficients. The following sample theorem should be compared with Theorem 1.

Theorem 8 ([34])**.** *Let $D(K)$ be a diagram of a hyperbolic link K, obtained as the closure of a positive braid with at least 3 crossings in each twist region. Then the state surface S_A satisfies*

$$\chi(\mathrm{guts}(S^3 \setminus S_A)) \;=\; \chi(\mathbb{G}'_A) \;=\; 1 - \left|\beta'\right|.$$

As a consequence, the volume of $S^3 \setminus K$ is bounded in terms of the Jones coefficient $|\beta'|$:

$$v_8 \left(\left|\beta'_K\right| - 1\right) \;\leq\; \mathrm{vol}(S^3 \setminus K) \;<\; 15 v_3 \left(\left|\beta'_K\right| - 1\right) - 10 v_3,$$

where $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

Similar volume estimates were proved for alternating links by Dasbach and Lin [22], and for several other families of links by Kalfagianni, Purcell, and myself [29, 30, 31, 34]. However, while previous estimates were all somewhat indirect, the lower bound on volume in Theorem 8 works by finding a direct and intrinsic relationship between the reduced state graph $\mathbb{G}'_A$ (hence, Jones coefficients) and the geometric topology of $S^3 \setminus K$. As a consequence of this direct argument, the upper and lower volume bounds differ by a rather small factor of about 4.15.

Theorems 7 and 8 are proved by constructing a detailed polyhedral decomposition of the surface complement $S^3 \setminus S_A$, whose combinatorial structure is closely linked to the combinatorics of the ribbon graph $\mathbb{G}_A$. In particular, all the I–bundles in the complement of S_A are visible in the individual polyhedra. As a result, we can detect when all of $S^3 \setminus S_A$ is an I–bundle (hence S_A is a fiber), as well as compute the guts.

4.3. **Ongoing project: extend these theorems to homogeneously adequate diagrams.** All of our structural results about the polyhedral decomposition of $S^3 \setminus S_\sigma$ apply equally well in the context where S_σ is the state surface of any adequate, homogeneous state σ. Thus we are able to prove theorems that detect fibers and guts in terms of the reduced state graph $\mathbb{G}'_\sigma$: see e.g. [24].

What's currently missing is a connection between state graphs and (colored) Jones polynomials, in the spirit of Theorem 5. We are currently pursuing an extension of this theorem to other states, using a gluing formula discovered by Armond and Dasbach.

An open question central to this line of inquiry is: does every knot K admit a diagram with an adequate, homogeneous state σ? While a "yes" answer seems too good to be true, no counterexamples are known. Knowing that the Jones and/or colored Jones polynomials have a particular form could be the key to constructing a counterexample. On the other hand, if the answer to this question is "yes," then results proved about these diagrams become particularly powerful.

4.4. **Future project: use the polyhedral decomposition to prove new results in geometric topology.** Our polyhedral decomposition of $S^3 \setminus S_\sigma$ for an adequate, homogeneous state σ generalizes Menasco's polyhedral decomposition of alternating links [52]. Menasco's decomposition has been fruitfully applied to prove results that involve incompressible surfaces: characterizing hyperbolic knots, ruling out exceptional surgeries, finding closed incompressible surfaces, etc. [47, 53, 54]. Results in this vein should also be feasible in our setting. Again, results in this vein become more powerful if the class of homogeneously adeqiate links is universal, or particularly large.

5. Angled triangulations

The geometric results described so far have all been coarse in nature: certain combinatorial measurements of a knot or a 3–manifold determine a geometric quantity (e.g. volume, geodesic length) up to a bounded error. By contrast, the method described in this section can often construct the full hyperbolic metric on a manifold from combinatorial data. At a minimum, the method of angled triangulations still provides a wealth of topological information for a very small cost.

5.1. **Constructing hyperbolic metrics.** A cusped 3–manifold M has a combinatorial description as a union of *ideal tetrahedra*. These are tetrahedra whose vertices have been removed; one can think of the vertices as lying on the boundary at infinity. To attempt to geometrize M, we can assign each tetrahedron a particular hyperbolic shape; these shapes are parametrized up to isometry by dihedral angles. In order to glue the tetrahedra coherently and get a complete hyperbolic structure on M, three conditions need to be satisfied:

(a) The dihedral angles around each edge must add up to 2π.
(b) There must not be any shearing of faces as we go around an edge.
(c) The ideal vertices must fit together in a way that keeps ∂M infinitely far away.

An *angle structure* on M is an assignment of dihedral angles to tetrahedra that only needs to satisfy condition (a). Because this condition is linear, an angle structure is a solution to a linear system of equations and inequalities (the inequalities keep the dihedral angles positive, and the tetrahedra convex). Thus the space of all angle structures is an open, bounded, convex polytope, making this type of structure relatively easy to obtain. By contrast, Mostow rigidity implies that a geometric gluing satisfying all of (a)–(c) must be unique.

By the work of Casson and Rivin, angle structures form a very useful stepping stone to finding a hyperbolic structure. Every angle structure has an associated *volume*, namely the sum of the volumes of all the tetrahedra. This defines a smooth, concave function on the polytope of solutions. Casson and Rivin showed that if the volume function has a critical point in this polytope, the corresponding tetrahedron shapes satisfy conditions (a)–(c), and give a hyperbolic metric [67, 26]. Thus to construct the hyperbolic metric, it suffices to show the open polytope contains a maximum of the volume function.

Guéritaud and I have carried out this full program for two families of manifolds with sufficiently tractable combinatorics [41]. Guéritaud worked out the case of punctured torus bundles over the circle. Following this lead, I extended the method to complements of two-bridge links. In addition to knowing how to build the hyperbolic metric, this method is very useful for estimates: because the complete structure comes from the *maximum* of the volume function, every angle structure gives a lower bound on hyperbolic volume. This led to particularly sharp volume bounds for both bundles and links.

5.2. **Ongoing project: extend the Casson–Rivin program to other families of manifolds.** The manifolds that fiber over the circle, as in Section 3, are natural candidates for this approach. Given a surface S and a pseudo–Anosov homeomorphism $\varphi : S \to S$, Agol has constructed a combinatorially canonical ideal triangulation of the mapping torus of φ [5]. His construction requires adding punctures to the surface, but is still extremely general. For Agol's *veering* triangulations, Guéritaud and I found an explicit construction of angle structures, and have partially parametrized the polytope of these structures [27].

The next step in our project is to complete the parametrization of the angle polytope. Once that is finished, we plan to analyze the behavior of the volume function, in a manner analogous to [41].

Our construction of angle structures for these triangulations is more than a partial result. A number of computational algorithms in 3–dimensional topology depend on enumerating normal surfaces in a triangulation. See *e.g.* [17, 49]. When the triangulation comes with angle structures, with an *a priori* lower bound on the smallest angle, the enumeration of normal surfaces up to a given genus becomes dramatically faster. Thus our work also aids in computational topology.

6. Gromov's surface subgroup conjecture

As part of his pioneering work on negatively curved groups, Gromov asked the following question. Let G be a word-hyperbolic group, which is not virtually free (that is, no finite-index subgroup of G is free). Does G necessarily contain the fundamental group of a surface? This question is wide open; in fact, it was only very recently that Kahn and Markovic showed the fundamental groups of hyperbolic n–manifolds contain surface subgroups [44].

In joint work with Thomas, I gave a positive answer to Gromov's question for lattices in hyperbolic buildings [38]. For parameters $p \geq 5$ and $g \geq 2$, *Bourdon's building* $I_{p,v}$ is a highly symmetric, negatively curved 2–complex, defined by the property that each 2–cell is a regular right-angled hyperbolic p–gon and the link at each vertex is the complete bipartite graph $K_{v,v}$. The isometry group $\mathrm{Aut}(I_{p,v})$ of such a building is uncountable, and remarkably rich. However, lattices in these isometry groups turn out to be fairly well-behaved.

Thomas and I went a long way toward characterizing the conditions under which a hyperbolic surface of genus g occurs as the quotient of $I_{p,v}$ under a lattice of isometries. In particular, we showed that when $p \geq 6$, every building $I_{p,v}$ covers a hyperbolic surface, which implies that every lattice in $\mathrm{Aut}(I_{p,v})$ contains a surface subgroup.

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