Projective Planes

Let $F$ be a field. Examples would include the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or finite fields $\mathbb{Z}_n$, where $n$ is prime\(^1\).

$V = F^3$ denotes the space of all triplets $(x, y, z)$ where the coordinates $x$, $y$, and $z \in F$. The projective plane of $F$, denoted $P(F)$ has points that are represented as one-dimensional subspaces of $V$ (as a vector space), and lines represented by two-dimensional subspaces. A point is on a line if, as a one-dimensional subspace it is contained in the line, as a two-dimensional subspace.

In the case $F = \mathbb{Z}_n$ the vector space $V$ has $n^3 - 1$ nonzero elements, and each point is a set containing $n - 1$ nonzero elements of $V$. The sets corresponding to distinct points intersect only at $(0,0,0)$. Thus $P(V)$ has $\frac{n^3 - 1}{n - 1}$ elements.

Exercise: Show that $P(V)$ satisfies all of the Fano axioms except the one that specifies the number of points on each line, unless $F = \mathbb{Z}_2$, in which case all the axioms hold.

\(^1\)Not all finite fields have this form