Crossbar Theorem

Theorem 1 Let \( X \) be a point interior to \( \angle BAC \). Then the ray \( \overrightarrow{AX} \) crosses the segment \( BC \).

![Diagram of Triangle ABC with a point X interior to \( \angle BAC \) and ray \( \overrightarrow{AX} \) crossing segment \( BC \).]

Proof Let a point \( D \) be chosen so that \( B - A - D \). Then we can apply the Pasch axiom to \( \triangle CDB \) and conclude that \( \overrightarrow{AX} \) crosses either \( BC \) or \( DC \).

Noting that \( D \) and \( B \) are on opposite sides of \( \overrightarrow{AC} \), it follows that \( D \) and \( X \) are on opposite sides of \( \overrightarrow{AC} \), and hence that all points of \( \overrightarrow{DC} \) are on the side of \( \overrightarrow{AC} \) opposite to all points on \( \overrightarrow{AX} \). Therefore \( \overrightarrow{AX} \) does not cross \( DC \). We must finally eliminate the possibility that the opposite ray, \( \overrightarrow{AY} \), crosses either \( DC \) or \( BC \). To do so we have only to notice that both of these segments are on the opposite side of \( \overrightarrow{DB} \) from \( \overrightarrow{AY} \).

The only possibility not eliminated is that \( \overrightarrow{AX} \) crosses \( BC \). \( \blacksquare \)