Preface

If an application of mathematics has a component that varies continuously as a function of time, then it probably involves a differential equation. For this reason, ordinary differential equations are of great importance in engineering, applied mathematics and the sciences. This has been recognized since the founders of calculus, Newton and Leibniz, made their contributions to the subject in the late seventeenth century.

A differential equations text has to address an audience with diverse interests. Science and engineering majors are required to take a differential equations course because it provides them with valuable mathematical tools. Mathematics majors may take courses in differential equations because the subject is interesting; because it is an essential component of applied mathematics; or because it is prerequisite for the study of differential geometry, dynamical systems, and mathematical modelling. All students who take a differential equations course will gain a deeper understanding of the concepts and applications of calculus.

This text is in four parts: the first introduces both linear differential equations and nonlinear systems, and provides a foundation. The second part is devoted to linear differential equations, including the single second order equation, systems of first order equations, Laplace transform methods, and equations with variable coefficients. The third part that focuses on nonlinear differential equations and dynamical systems. The fourth part of the text, on boundary value problems, focuses on applications of the technique of separation of variables for partial differential equations. The dependency graph that follows the table of contents will be of assistance in navigating the text and in planning class syllabi. In this graph, the sections marked with asterisks in the table of contents are treated as separate nodes. They can be skipped without interruption of the sequence within the chapters where they reside. In a recent course at Temple University, given over a semester of fourteen weeks (not including the final examination), I covered chapters 1–6, and section 8.1, skipping sections 1.5, 2.5, 3.4, 3.5, 5.9, and 6.8. The class met three hours per week, with an additional weekly Maple-based computer laboratory.

To narrow the field when selecting a differential equations text, an instructor will ask if the applications are realistic, if linear algebra is a prerequisite, and if the level of rigor is appropriate for his or her students. This text uses the traditional toy problems rather than ones involving “real data.” While there is much to be said for studying realistic applications, the many complications may obscure the differential equation that should be the center of attention. It is usually preferable to consider a simple problem that exemplifies the differential equations component of an actual application.

Since linear differential equations are the central topic in most ordinary dif-
Differential equations courses, it would be logical to include linear algebra as a prerequisite. After all, there is essentially nothing in the syllabus of an introductory linear algebra course that can’t be put to use in a differential equations course. In spite of this, many universities, including the one that employs me, require only two semesters of calculus as a prerequisite for their differential equations courses. While I would recommend taking linear algebra before differential equations to any student, I have written this text with the intention of accommodating the needs of readers who have not had a linear algebra course. Chapter 4 does present linear systems of differential equations in matrix form. Since our work is with systems of two equations — and thus involves only $2 \times 2$ matrices — the time required to make it accessible to those who have not had a linear algebra course is not excessive. Linear operators are defined and discussed in chapter 5, along with further concepts from linear algebra: linear combinations and linear independence.

This text maintains a moderate level of rigor. Proofs are included if they are accessible and have the potential to enhance the reader’s understanding of the subject. For example, the existence theorem for solutions of initial value problems is not actually proved — its prerequisite, Ascoli’s theorem, would not ring a bell for most readers — but I allude to Peano’s proof, using Euler’s method to show that a solution exists. On the other hand, the proof of the uniqueness theorem is included, as a special case of proposition 2.4.3, which specifies an upper bound for the rate at which solutions of a differential equation can diverge from one another.

I have used an approach to definitions that is more common in lower level mathematics texts. Instead of placing formal, numbered statements of definitions in the text, I have provided each chapter with a glossary that contains all of the definitions, in alphabetical order. This allows for a more informal discussion of a term when it is introduced. The first use of a term that is defined in the glossary is in boldface. I have reserved the bold typeface for that purpose — except when it is used in mathematical expressions. My hope is that readers, who are accustomed to hypertext, will find this approach to be convenient.

Confronting a solution of a differential equation that was not obtained by symbol manipulation is for many people a first encounter with a function that is not presented as a formula. While no student should pass a differential equations course without learning how to solve certain differential equations by analytic means, we should also expect students to be able to use numerical methods effectively. My goal has been to induce an intuitive understanding of the concept of a solution of an initial value problem in order to resolve potential confusion about what we are approximating when we call upon a numerical method. I have not attempted to duplicate the contents of a numerical analysis text. The discussion of numerics is confined to Euler’s method — which advances the understanding of what a differential equation is — and a brief “user’s guide” to more advanced methods. I do not
hesitate to call upon effective numerical methods when symbolic methods can’t be
used.

The existence and uniqueness theorems for initial value problems have implications about the structure of the set of solutions of a differential equation or system. The text explores that structure, as well as the additional structural properties that special kinds of equations, such as linear equations or autonomous systems have. The properties of linearity and of being autonomous are important for many applications, and by associating these properties with particular applications, we can bring our physical experience to the task of learning about differential equations.

It is widely believed that computer algebra software (CAS) can make short work of the routine calculations that have bedeviled generations of students in introductory differential equations courses. I have found that this software is sometimes beneficial as a labor saving device, and that it is definitely useful for producing illustrations. There are three admirable CAS programs available: Maple, Mathematica, and Matlab. Most of the illustrations were produced with Mathematica. Examples using CAS programs can be found on the text web site, http://www.prenticehallmath.com/conrad. This text can be used effectively without the benefit of CAS, but an IVP solver — a computer program that will display graphs of solutions of differential equations on a computer or calculator screen — is required. Every CAS can function as an IVP solver, and special-purpose IVP solvers may be downloaded by following links on the text web site. The web site also has a list of currently available hand held graphing calculators that include IVP solvers. There is a general discussion of IVP solvers in section 3.3.

Part I is divided into three chapters, covering linear first order equations, nonlinear first order equations, and systems of first order equations, respectively. The goal of chapter 1 is to present linear equations and a few applications (linear growth and mixture problems) that illuminate the meaning of linearity. The method of solution is variation of constants, rather than the Leibniz integrating factor, because it leads to a discussion of the structure of the general solution as the sum of a particular solution and the associated homogeneous solution. The chapter includes an introduction to initial value problems, with a careful explanation of the existence and uniqueness theorems.

Chapter 2 opens by visualizing differential equations with the aid of a direction field, and this leads to Euler’s method. We then turn to the symbolic algorithms for solving separable and exact equations. We notice that these algorithms do not actually produce an explicit solution as the algorithm for solving linear equations did; instead, the output is an integral: a function \( F(x, y) \) whose level curves define solutions. The theoretical component of the chapter is the existence and uniqueness theorems, and the applications are to nonlinear growth and falling bodies with air drag.
Systems of differential equations traditionally make their appearance later in a text than they do here. Since systems do play a prominent role in applications, and there are many elementary things to be said about them, I have introduced them in part I. A solution of a system of two first order differential equations is viewed as a pair of parametric equations for a curve in the phase plane. If the system is autonomous, it can be visualized as a vector field on the phase plane. The method for finding an integral of a system of two autonomous equations is presented, and to make accurate graphs of solutions, there is an informal discussion of IVP solvers — the “user’s guide” that was mentioned above. Applications are to the van der Pol equation and to population models for two interacting species.

Part II is devoted to linear differential equations, starting with systems. Practically all of the systems in chapter 4 are of two equations with constant coefficients, stated in explicit form. The solution algorithm is invariably to compute the characteristic roots and vectors of the coefficient matrix, determine a fundamental matrix solution, and in the inhomogeneous case, to use variation of constants to determine a particular solution. This algorithm elucidates the structure of the solution set, by enabling us to sketch phase portraits for homogeneous systems, and to identify centers, nodes and saddles readily (these geometric aspects of systems are to be found in section 8.1, which can be read just after section 4.3). Readers are expected to learn to find characteristic roots and vectors of \( 2 \times 2 \) matrices, as well as the definition of characteristic roots and vectors of larger matrices. Chapter 4 contains an introduction to matrices (concentrating on the \( 2 \times 2 \) case), and a review of the complex number system.

The centerpiece of the traditional first course in ordinary differential equations is analysis of second order linear equations. By placing this topic after linear systems, we realize an economy, because second order equations are easily transformed into equivalent systems, and the results of chapter 4 can be applied. The connection with systems is exploited to streamline the presentation of the theoretical aspects of second order equations, but the solution algorithm for equations with constant coefficients is presented in the usual way. In chapter 5, I have emphasized the vector space structure of the set of solutions of a linear homogeneous differential equation and the concept of a linear differential operator. About half of the chapter is devoted to an application, damped mechanical systems with one degree of freedom. I have not used this application to convince readers that differential equations is applicable mathematics, but to enable visualization of the concepts presented. The important properties of second order linear differential equations can be better presented in terms of mechanical systems than in any other context. The chapter also includes a discussion of mechanical systems with two degrees of freedom, which are modelled by systems of two second order equations. The emphasis is on finding fundamental frequencies and modes of vibration.
The Laplace transform is presented in chapter 6 as an alternate means of solving inhomogeneous linear equations or systems with constant coefficients. The goal is to understand how transform methods work. It is natural to use the Laplace transform to motivate and explore the convolution, and this leads to a different interpretation of the variation of constants formula that was studied in chapter 5. A second goal of this chapter is to introduce the Heaviside unit step function and its “derivative,” the Dirac delta function. The applications in chapter 6 are primarily to electrical circuits.

Chapter 7 is devoted to series solutions of linear differential equations with variable coefficients. The reason to study series solutions is to express solutions of linear differential equations with variable coefficients in terms of elementary and special functions. The expansion of a solution near a regular singular point is considered in some detail, as a generalization of the solution of Cauchy-Euler equations. Thus, Bessel functions can be introduced, and we use them in chapter 11 to explore the solution of the diffusion equation in a cylindrical domain.

Nonlinear differential equations are the subject of part III, which can be read at any time after chapter 4. Chapter 8 is about the stability and asymptotic stability of stationary points of autonomous nonlinear systems. Stability is analyzed by means of linearization and by the method of Lyapunov. A detailed understanding of phase portraits and the stability of linear systems is an essential prerequisite for comprehending linearization, and these topics are covered in the first two sections of chapter 8. Instructors who emphasize the geometric aspect of systems will want to cover these sections in the midst of chapter 4.

Chapter 9 asks why it is that two dimensional autonomous systems are so simple, and three dimensional systems so complicated? Our analysis of this question leads us to consider Poincaré mappings. The simplest nontrivial example occurs in the case of systems of linear equations with periodic coefficients, and the theorem of Floquet. The Poincaré mapping associated with a system of two autonomous differential equations is a one-dimensional dynamical system, while a three-dimensional system of differential equations reduces to a dynamical system on the plane. To answer the question posed by chapter 9 we therefore turn to a comparison of one- and two-dimensional dynamical systems. The chapter concludes with a heuristic study of the forced Duffing equation that attempts to tie together all of the threads of part III.

Part IV starts with the pure two-point boundary value problem. The variation of constants formula reappears in the guise of Green’s function, which is constructed as a way of exploring existence and uniqueness of solutions of two-point boundary value problems. The application here is to a mechanical system, for which data on position or velocity are obtained at two different times. After section 10.1, our attention will be focused on boundary problems that do not have unique solutions.
The initial motivation for such problems is to understand why the existence and uniqueness of solutions, which we take for granted with initial value problems, are not available in other venues. What is more surprising is that with boundary value problems, the situations where uniqueness fails are of more interest in applications than the situations where uniqueness holds. We turn nonuniqueness to our advantage by studying the classical partial differential equations of mathematical physics — the initial-boundary value problems for the diffusion and wave equations, and the three boundary value problems for the Laplace equation — by the method of separation of variables. The basic technique, separation of variables, is presented in chapter 10 within the context of the diffusion equation. This leads to a brief introduction to Fourier sine series. The other boundary value problems are postponed for coverage in chapter 11. Chapter 10 concludes with the Sturm oscillation theorems, providing yet another opportunity for phase plane analysis of second order linear differential equations.

Chapter 11 brings the results of chapter 7 on series solutions, and chapter 10, on separation of variables, to bear upon the classical partial differential equations of mathematical physics: the diffusion, wave, and Laplace equations. Bessel functions and Legendre polynomials, which were introduced in chapter 7, are used to express solutions on domains with particular kinds of symmetry.