

Modular Forms Representable As Eta Products

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ABSTRACT

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In this dissertation, we discuss modular forms that are representable as eta products and generalized eta products. Eta products appear in many areas of mathematics in which algebra and analysis overlap. M. Newman [15, 16] published a pair of well-known papers aimed at using eta-product to construct forms on the group $\Gamma_0(n)$ with the trivial multiplier system. Our work here divides into three related areas. The first builds upon the work of Siegel [23] and Rademacher [19] to derive modular transformation laws for functions defined as eta products (and related products). The second continues work of Kohnen and Mason [9] that shows that, under suitable conditions, a generalized modular form is an eta product or generalized eta product and thus a classical modular form. The third part of the dissertation applies generalized eta-products to rederive some arithmetic identities of H. Farkas [5, 6].

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CHAPTER 1

Introduction

1.1 Basic Definitions

By $SL_2(\mathbb{Z})$ we mean the group of 2×2 matrices with integral entries and determinant 1. We call $SL_2(\mathbb{Z})$ the modular group $\Gamma(1)$. We define the action of an element $A \in SL_2(\mathbb{Z})$ on the upper half plane H by

$$Az = \frac{az + b}{cz + d},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For a positive integer N , we define a subgroup of $\Gamma(1)$;

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, b \equiv c \equiv 0 \pmod{N}, a \equiv d \equiv 1 \pmod{N}, ad - bc = 1 \right\}.$$

We call this subgroup the *principal congruence subgroup of level N* . For any other subgroup $\Gamma \subset \Gamma(1)$, if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{Z}$, then we call Γ a *congruence subgroup*.

Definition 1.1 Let Γ be a subgroup of $\Gamma(1)$. A *fundamental region* for Γ is an open subset R of H such that

1. no two distinct points of R are equivalent with respect to Γ , and
2. every point of H is equivalent to some point in the closure of R .

Proposition 1.1 *The full modular group is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and*

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Definition 1.2 *We say that ν is a multiplier system for the group Γ and weight k provided $\nu(M)$, $M \in \Gamma$, is a complex-valued function of absolute value 1, satisfying the following equation*

$$\nu(M_1 M_2)(c_3 \tau + d_3)^k = \nu(M_1)\nu(M_2)(c_1 M_2 \tau + d_1)^k (c_2 \tau + d_2)^k,$$

where $M_1 = \begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}$ and $M_3 = M_1 M_2 = \begin{pmatrix} * & * \\ c_3 & d_3 \end{pmatrix}$.

Definition 1.3 *Let R be a fundamental region of Γ . A parabolic point (or parabolic vertex or parabolic cusp) of Γ is any real point q , or $q = \infty$, such that $q \in \text{closure}(R)$, in the topology of the Riemann sphere.*

Definition 1.4 *Suppose $\Gamma \subset \Gamma(1)$ such that $[\Gamma(1) : \Gamma] = \mu$. Let A_1, A_2, \dots, A_μ be a set of right coset representatives of Γ in $\Gamma(1)$. The width of Γ at $q_j = \infty$ is the smallest positive integer λ such that $S^\lambda \in \Gamma$. Also, the width of Γ at any other cusp $q_j = A_j(\infty)$ is the smallest positive integer λ such that $A_j S^\lambda A_j^{-1} \in \Gamma$.*

Definition 1.5 *Let $k \in \mathbb{R}$ and $\nu(M)$ a multiplier system for Γ and of weight k . A function $F(\tau)$ defined and meromorphic in H is a modular form (MF) of weight k , with multiplier system (MS) ν , with respect to Γ , provided*

1. $F(M\tau) = \nu(M)(c\tau + d)^k F(\tau)$

for every $M \in \Gamma$;

2. The Fourier expansion of F at every cusp q_j has the form

$$F(\tau) = \sigma_j(\tau) \sum_{n=-n_0}^{\infty} a_n(j) e^{2\pi i(n+\kappa_j)(A_j^{-1}\tau)/\lambda_j},$$

where

$$\begin{aligned}\sigma_j(\tau) &= 1 \quad \text{if} \quad q_j = \infty \\ \sigma_j(\tau) &= (\tau - q_j)^{-k} \quad \text{if} \quad q_j \text{ is finite.}\end{aligned}$$

Here λ_j is the width at q_j

and $0 \leq \kappa_j < 1$ is determined by

$$\nu(AS^{\lambda_j}A^{-1}) = e^{2\pi i\kappa_j},$$

if q_j is finite and

$$\nu(S^{\lambda_j}) = e^{2\pi i\kappa_j},$$

if q_j is infinity.

If the first nonzero $a_n(j)$ occurs for $n = -n_0 < 0$, we say F has a pole at q_j of order $n_0 - \kappa_j$. If the first nonzero $a_n(j)$ occurs for $n = n_0 \geq 0$, we say F is regular at q_j with a zero of order $n_0 + \kappa_j$.

To decide whether a given function is a modular form on Γ , it is essential to determine how this function transforms under the action of Γ . With respect to the full modular group, it will be enough to determine how the function transforms under the generators S and T . Usually, it is easier to see how the function transforms under the action of S . In Chapter 2 of this thesis, we determine the transformation law of $\theta_3(w, \tau)$ under the action of T using Siegel's method [23]. Notice that $\theta_3(w, \tau)$ is not a modular form but $\theta_3(0, \tau) = \theta_3(\tau)$ is. We will see in the (2.1.1) that $\theta_3(\tau)$ is a modular form of weight $\frac{1}{2}$. We then generalize Siegel's method to determine the transformation laws for an entire class of modular forms under $\Gamma_0(N)$. Here $\Gamma_0(N)$ is a congruence subgroup to be defined later. This class of functions is a product of eta functions with very important properties to be used in the later chapters. We then impose some conditions to derive a class of functions which is invariant under $\Gamma_0(N)$. These kinds of invariant functions were first constructed by Newman [15, 16].

In Chapter 3, we define *generalized modular forms* and present some theorems derived by Kohnen and Mason. In [9], they impose some conditions upon

the order of the function at the cusps and prove that the generalized modular form is representable as an eta product in the form described in Chapter 3. I present another class of functions which generalizes the class presented in Chapter 3. These well-known functions are called *generalized eta products* [20, 21]. We relax the condition of Kohnen and Mason from a condition on the order of the function at the cusps to a condition on the level of the congruence subgroup. We then deduce some results on other congruence subgroups. As a result, we represent *generalized modular forms* as generalized eta products.

In Chapter 4, we use the fact that the logarithmic derivative of the generalized eta products will span the space of $M_2(\Gamma_0(4))$ and determine some arithmetic identities modulo 4 by relating the logarithmic derivative of the generalized eta functions to Eisenstein series of weight 2. We also determine arithmetic identities modulo the primes 3 and 7.

CHAPTER 2

Transformation Laws Of Classes Of Functions

2.1 Transformation Law of Jacobi $\theta_3(w, \tau)$

Let w be a complex number. The function $\theta_3(w, \tau)$ is defined by

$$\theta_3(w, \tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2w + q^{4n-2}) , \quad (2.1)$$

where $q = e^{\pi i \tau}$ and τ is in the upper half plane [25]. The transformation law is given by

$$\theta_3\left(\frac{w}{\tau}, \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{-\frac{w^2}{\pi i \tau}} \theta_3(w, \tau) , \quad (2.2)$$

We give a new, detailed proof using Residue Calculus inspired by Siegel's proof of the transformation law of the Dedekind eta function [23]. First, we prove (2.2) for $\tau = iy$, where $w = \sigma + it$ and $y > \frac{2|t|}{\pi}$, and then extend the result to all τ in the upper half plane by analytic continuation.

We use the logarithmic expansion to prove (2.2). In proving the transformation law of the logarithmic derivative, we will encounter some problems with the zeroes of the theta function. The zeroes of $\theta_3(w, \tau)$ are the points $w = \frac{\pi}{2} + \frac{\pi \tau}{2} + m\pi + n\pi\tau$, for $m, n \in \mathbb{Z}$.

To solve this problem, we first fix w such that $\text{Re} w \neq \frac{\pi}{2} + n\pi$, and prove the

transformation law for $\tau = iy$. We then extend the result by analytic continuation to the whole τ plane. Once we have it for all w such that $\operatorname{Re} w \neq \frac{\pi}{2} + n\pi$, we use analytic continuation in the w plane to extend the result to all w .

Theorem 2.1 *If $\tau = iy$ and $y > \frac{2|t|}{\pi}$, where $w = \sigma + it$, then $\theta_3(w, \tau)$ satisfies*

$$\theta_3\left(\frac{w}{iy}, \frac{i}{y}\right) = (y)^{\frac{1}{2}} e^{\frac{w^2}{\pi y}} \theta_3(w, iy) . \quad (2.3)$$

Proof Fix w such that $\operatorname{Re} w \neq \frac{\pi}{2} + n\pi$. Then it is sufficient to prove

$$\log \theta_3(w, iy) - \log \theta_3\left(\frac{w}{iy}, \frac{i}{y}\right) + \frac{w^2}{\pi y} = -\frac{1}{2} \log y . \quad (2.4)$$

If we simplify $\theta_3(w, \tau)$, we get

$$\begin{aligned} \theta_3(w, \tau) &= \prod_{n=1}^{\infty} (1 - q^{2n})(e^{2iw} + q^{2n-1})(e^{-2iw} + q^{2n-1}) \\ &= \prod_{n=1}^{\infty} (1 - q^{2n}) \left(1 + \frac{q^{2n-1}}{e^{2iw}}\right) \left(1 + \frac{q^{2n-1}}{e^{-2iw}}\right). \end{aligned}$$

Since $y > \frac{2|t|}{\pi}$, $\left| \frac{q^{2n-1}}{e^{2i\omega}} \right| < 1$. Thus the expansion of $\log \theta_3(w, iy)$ is

$$\begin{aligned}
\log \theta_3(w, iy) &= \sum_{n=1}^{\infty} \log(1 - q^{2n}) + \sum_{n=1}^{\infty} \log\left(1 + \frac{q^{2n-1}}{e^{2i\omega}}\right) + \sum_{n=1}^{\infty} \log\left(1 + \frac{q^{2n-1}}{e^{-2i\omega}}\right) \\
&= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^{2n})^m}{m} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} (q^{2n-1})^m}{m e^{2im\omega}} \\
&\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} (q^{2n-1})^m}{m e^{-2im\omega}} \\
&= -\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{q^{2m}}{1 - q^{2m}} \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m e^{2im\omega}} q^{-m} \left(\frac{q^{2m}}{1 - q^{2m}} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m e^{-2im\omega}} q^{-m} \left(\frac{q^{2m}}{1 - q^{2m}} \right) \\
&= -\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{e^{-2\pi y m}}{1 - e^{-2\pi y m}} \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \frac{e^{\pi m y}}{e^{2im\omega}} \left(\frac{e^{-2\pi y m}}{1 - e^{-2\pi y m}} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \frac{e^{\pi m y}}{e^{-2im\omega}} \left(\frac{e^{-2\pi y m}}{1 - e^{-2\pi y m}} \right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{1 - e^{2\pi y m}} \right) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2i\omega m} \left(\frac{e^{\pi y m}}{1 - e^{2\pi y m}} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2i\omega m} \left(\frac{e^{\pi y m}}{1 - e^{2\pi y m}} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\log \theta_3\left(\frac{w}{iy}, \frac{i}{y}\right) &= \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{1 - e^{2\pi m/y}} \right) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2m\omega/y} \left(\frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2m\omega/y} \left(\frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} \right).
\end{aligned}$$

So we have to prove that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{1 - e^{2\pi y m}} \right) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2i\omega m} \left(\frac{e^{\pi y m}}{1 - e^{2\pi y m}} \right) \\
&+ \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2i\omega m} \left(\frac{e^{\pi y m}}{1 - e^{2\pi y m}} \right) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{1 - e^{2\pi m/y}} \right) \\
&- \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2m\omega/y} \left(\frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} \right) - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2m\omega/y} \left(\frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} \right) + \frac{w^2}{\pi y} = -\frac{1}{2} \log y.
\end{aligned}$$

To prove this, consider

$$F_n(z) = -\frac{1}{8z} \cot \pi i N z \cot \pi N z / y + \frac{1}{z} \left(\frac{e^{-iNz(\pi/y+2w/y+2w)}}{1 - e^{-2\pi i N z / y}} \right) \left(\frac{e^{Nz(\pi+2iw)}}{1 - e^{2\pi z N}} \right),$$

where $N = n + \frac{1}{2}$.

We will calculate the residues of $F_n(z)$ at the poles $z = 0$, $z = \frac{ik}{N}$ and $z = \frac{ky}{N}$ for $k = \pm 1, \pm 2, \dots, \pm n$.

We start by calculating the residue of $F_n(z)$ at $z = 0$. We use Bernoulli numbers to calculate the residue of the second part of the function. The residue at 0 of the first summand of the function is $\frac{i}{24} \left(y - \frac{1}{y} \right)$. Now for the second summand of the function we will use the fact that

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad (2.5)$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$.

Notice that

$$\frac{1}{z} \left(\frac{e^{-iNz(\pi/y+2w/y+2w)}}{1 - e^{-2\pi i N z / y}} \right) \left(\frac{e^{Nz(\pi+2iw)}}{1 - e^{2\pi z N}} \right) = \frac{-y}{4\pi^2 i N^2 z^3} \left(\frac{-2\pi i N z / y}{e^{-2\pi i N z / y} - 1} \right) \left(\frac{2\pi N z}{e^{2\pi z N} - 1} \right) \\ * e^{Nz(-i\pi/y-2iw/y+\pi)}.$$

Using (2.5) and the Taylor expansion of the exponential function, we see easily that the residue at $z = 0$ of the second summand of the function is

$$\frac{-y}{4\pi^2 i N^2} \left\{ \frac{N^2}{2} \left(-\frac{\pi^2}{y^2} - \frac{4w^2}{y^2} + \pi^2 - \frac{4\pi w}{y^2} - \frac{2\pi^2 i}{y} - \frac{4\pi i w}{y} \right) \right. \\ \left. + \left(-\frac{\pi^2 N^2 i}{y} - \frac{\pi^2 N^2}{3y^2} + \frac{\pi^2 N^2}{3} \right) + \left(\frac{\pi i N^2}{y} - \pi N^2 \right) \left(-\frac{i\pi}{y} - \frac{2iw}{y} + \pi \right) \right\}.$$

Simplifying the above result, we conclude that the residue of the second summand at $z = 0$ is

$$\frac{w^2}{2\pi^2 i y} - \frac{i}{24} \left(y - \frac{1}{y} \right).$$

As a result we obtain

$$\text{Res}[F_n(z), 0] = \frac{w^2}{2\pi^2 i y}.$$

We note that

$$\operatorname{Res}[F_n(z), \frac{ik}{N}] = \frac{1}{8\pi k} \cot \frac{\pi ik}{y} - \frac{(-1)^k}{2\pi ik} e^{2kw/y} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}}.$$

Thus,

$$\begin{aligned} \sum_{\substack{k=-n; k \neq 0 \\ z = \frac{ik}{N}}}^n \operatorname{Res} [F_n(z), \frac{ik}{N}] &= 2 \sum_{k=1}^n \frac{1}{8\pi k} \cot \frac{\pi ik}{y} - \sum_{k=1}^n \frac{(-1)^k}{2\pi ik} e^{2kw/y} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \\ &\quad - \sum_{k=1}^n \frac{(-1)^k}{2\pi ik} e^{-2kw/y} \frac{e^{-\pi k/y}}{1 - e^{2\pi k/y}} \\ &= \frac{1}{4\pi i} \sum_{k=1}^n \frac{1}{k} - \frac{1}{2\pi i} \sum_{k=1}^n \frac{1}{k(1 - e^{2\pi k/y})} \\ &\quad - \frac{1}{2\pi i} \sum_{k=1}^n \frac{(-1)^k}{k} e^{2kw/y} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \\ &\quad - \frac{1}{2\pi i} \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2kw/y} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}}. \end{aligned}$$

The residue of $F_n(z)$ at $z = \frac{ky}{N}$ is

$$\operatorname{Res}[F_n(z), \frac{ky}{N}] = -\frac{1}{8\pi k} \cot \pi iky + \frac{(-1)^k}{2\pi ik} e^{-2ikw} \frac{e^{\pi ky}}{1 - e^{2\pi ky}}.$$

Thus,

$$\begin{aligned} \sum_{\substack{k=-n; k \neq 0 \\ z = \frac{ky}{N}}}^n \operatorname{Res} [F_n(z), \frac{ky}{N}] &= 2 \sum_{k=1}^n -\frac{1}{8\pi k} \cot \pi iky + \sum_{k=1}^n \frac{(-1)^k}{2\pi ik} e^{-2ikw} \frac{e^{\pi ky}}{1 - e^{2\pi ky}} \\ &\quad + \sum_{k=1}^n \frac{(-1)^k}{2\pi ik} e^{2ikw} \frac{e^{\pi ky}}{1 - e^{2\pi ky}} \\ &= -\frac{1}{4\pi i} \sum_{k=1}^n \frac{1}{k} + \frac{1}{2\pi i} \sum_{k=1}^n \frac{1}{k(1 - e^{2\pi ky})} \\ &\quad + \frac{1}{2\pi i} \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2ikw} \frac{e^{\pi ky}}{1 - e^{2\pi ky}} + \frac{1}{2\pi i} \sum_{k=1}^n \frac{(-1)^k}{k} e^{2ikw} \frac{e^{\pi ky}}{1 - e^{2\pi ky}}. \end{aligned}$$

Thus,

$$\begin{aligned}
2\pi i \sum_{\substack{k=-n \\ z=\frac{ky}{N}; z=\frac{ik}{N}}}^n \operatorname{Res} F_n(z) &= \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{1 - e^{2\pi y k}} \right) + \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2i w k} \left(\frac{e^{\pi y k}}{1 - e^{2\pi y k}} \right) \\
&+ \sum_{k=1}^n \frac{(-1)^k}{k} e^{2i w k} \left(\frac{e^{\pi y k}}{1 - e^{2\pi y k}} \right) - \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{1 - e^{2\pi k/y}} \right) \\
&- \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2k w/y} \left(\frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \right) \\
&- \sum_{k=1}^n \frac{(-1)^k}{k} e^{2w k/y} \left(\frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \right) + \frac{w^2}{\pi y}.
\end{aligned}$$

It remains to prove that

$$\lim_{n \rightarrow \infty} \oint_C F_n(z) dz = -\frac{1}{2} \log y,$$

where C is the parallelogram of vertices $y, i, -y$ and $-i$ taken counterclockwise.

Now it is easy to see that $\lim_{n \rightarrow \infty} z F_n(z)$ is $\frac{1}{8}$ on the edges connecting y to i and $-y$ to $-i$ and the limit $-\frac{1}{8}$ on the other two edges. Moreover, $F_n(z)$ is uniformly bounded on C for all n . Hence by the bounded convergence theorem we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \oint_C F_n(z) dz &= \oint_C z F_n(z) \frac{dz}{z} \\
&= \frac{1}{8} \left[- \int_{-i}^y \frac{dz}{z} + \int_y^i \frac{dz}{z} - \int_i^{-y} \frac{dz}{z} + \int_{-y}^{-i} \frac{dz}{z} \right] \\
&= \frac{1}{4} \left[- \int_{-i}^y \frac{dz}{z} + \int_y^i \frac{dz}{z} \right] \\
&= \frac{1}{4} \left[- \left(\log y + \frac{\pi i}{2} \right) + \left(\frac{\pi i}{2} - \log y \right) \right] \\
&= -\frac{1}{2} \log y.
\end{aligned}$$

This completes the proof.

2.1.1 The Jacobi function $\theta_3(\tau)$

At $w = 0$, we have

$$\theta_3(\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2$$

The transformation law is given by

$$\theta_3\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \theta_3(\tau) . \quad (2.6)$$

To obtain (2.6) simply set $w = 0$ in (2.3).

2.1.2 The Function $w(z, \tau)$

Let

$$w(z, \tau) = e^{\frac{-\pi iz}{12}} \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2z + q^{4n-2}).$$

Using the same technique but with different $F_n(z)$, we will be able to prove that $w(\tau)$ is invariant under the transformation $\tau \rightarrow \frac{-1}{\tau}$. In other words, by defining a suitable $F_n(z)$, we will be able to prove that $w\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = w(z, \tau)$. Following exactly the same steps in proving the transformation law for $\operatorname{Re} z \neq \frac{\pi}{2} + n\pi$ where $\tau = iy$ and then using analytic continuation to extend the result, we find that it is sufficient to prove that $\log w(z, \tau) - \log w\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = 0$ for $\tau = iy$ and $\operatorname{Re} z \neq \frac{\pi}{2} + n\pi$. Using logarithmic expansion we see that

$$\log w(z, iy) = \frac{\pi y}{12} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2imz} \frac{e^{\pi y m}}{1 - e^{2\pi y m}} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2imz} \frac{e^{\pi y m}}{1 - e^{2\pi y m}}.$$

So we have to prove that

$$\begin{aligned} & \frac{\pi}{12} \left(y - \frac{1}{y} \right) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2imz} \frac{e^{\pi y m}}{1 - e^{2\pi y m}} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2imz} \frac{e^{\pi y m}}{1 - e^{2\pi y m}} \\ & - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2imz} \frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{-2imz} \frac{e^{\pi m/y}}{1 - e^{2\pi m/y}} = 0. \end{aligned}$$

Consider

$$F_n(z) = \frac{1}{z} \left(\frac{e^{-iNz(\pi/y+2w/y+2w)}}{1 - e^{-2\pi iNz/y}} \right) \left(\frac{e^{Nz(\pi+2iw)}}{1 - e^{2\pi zN}} \right)$$

where $N = n + \frac{1}{2}$.

We repeat the process, calculating the residues of the poles of $F_n(z)$ at $z = 0$, $z = \frac{ik}{N}$ and at $z = \frac{ky}{N}$. As a result, we get

$$\begin{aligned} 2\pi i \sum_{k=1}^n \operatorname{Res}[F_n(z), \frac{ik}{N}] &+ 2\pi i \sum_{k=1}^n \operatorname{Res}[F_n(z), \frac{ky}{N}] \\ &= \frac{\pi}{12} \left(y - \frac{1}{y} \right) + \sum_{k=1}^n \frac{(-1)^k}{k} e^{2ikz} \frac{e^{\pi y k}}{1 - e^{2\pi y k}} \\ &+ \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2ikz} \frac{e^{\pi y k}}{1 - e^{2\pi y k}} - \sum_{k=1}^n \frac{(-1)^k}{k} e^{2ikz} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \\ &- \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2ikz} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}}. \end{aligned}$$

It is also easy to show that

$$\lim_{n \rightarrow \infty} z F_n(z) = 0.$$

on all the edges of the parallelogram connecting y to i , i to $-y$, $-y$ to $-i$ and $-i$ to y . We also see that $F_n z$ is uniformly bounded on C , then by the bounded convergence theorem, we get

$$\lim_{n \rightarrow \infty} \oint_C F_n(z) dz = 0.$$

By the Residue Theorem, we get

$$\begin{aligned} \frac{\pi}{12} \left(y - \frac{1}{y} \right) &+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{2ikz} \frac{e^{\pi y k}}{1 - e^{2\pi y k}} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-2ikz} \frac{e^{\pi y k}}{1 - e^{2\pi y k}} \\ &- \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{2ikz} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-2ikz} \frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} = 0. \end{aligned}$$

As a result, we get $w\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = w(z, \tau)$

The Function $w_1(\tau)$

Letting $z = 0$ in $w(z, \tau)$, we get

$$w(\tau) = e^{\frac{-\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 + q^{2n-1})^2.$$

Define $w_1(\tau) = \sqrt{w(\tau)} = e^{-\frac{\pi i \tau}{24}} \prod_{n=1}^{\infty} (1 + q^{2n-1})$. Notice that $w_1(\tau) \neq 0$ in the upper half plane. Thus by the transformation law, $w_1(\frac{-1}{\tau}) = \pm w_1(\tau)$. By setting $\tau = i$, we obtain

$$w_1\left(\frac{-1}{\tau}\right) = w_1(\tau).$$

2.2 Transformation Laws of a Class of Eta Products

Let τ be in the upper half plane and $n \in \mathbb{Z}$. The Dedekind eta function is defined by,

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Consider

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, c \equiv 0 \pmod{n}, ad - bc = 1 \right\},$$

a congruence subgroup of the full modular group.

Suppose $n > 1$, and let $\{r_\delta\}$ and $\{r'_\delta\}$ be two sequences of positive integers indexed by the positive divisors δ of n and suppose that n has g divisors.

Consider the function

$$g_1 = g_1(\tau) = \prod_{l=1}^g \frac{\eta(\delta_l \tau)^{r_{\delta_l}}}{\eta(\tau)^{r'_{\delta_l}}}.$$

We prove the transformation law of this function which is given by

$$g_1(V\tau) = e^{-\pi i \delta^*} \{-i(c\tau + d)\}^{\frac{1}{2} \sum_{l=1}^g r_{\delta_l} - \frac{1}{2} \sum_{l=1}^g r'_{\delta_l}} g_1(\tau),$$

where

$$\delta^* = \sum_{l=1}^g \left\{ \frac{a+d}{12c} + s(-d, c) \right\} r'_{\delta_l} - \sum_{l=1}^g \left\{ \frac{a+d}{12c_l} + s(-d, c_l) \right\} r_{\delta_l},$$

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left[\frac{hr}{k} \right] - \frac{1}{2} \right),$$

$c = c_l \delta_l$, and $V \in \Gamma_0(n)$.

A special case of the above product is given by

$$f(\tau) = \prod_{l=1}^g \left(\frac{\eta(\delta_l \tau)}{\eta(\tau)} \right)^{r_{\delta_l}}.$$

if we set $r_{\delta} = r'_{\delta}$ for every δ dividing n in $g(\tau)$.

Imposing certain conditions on r_{δ} 's and r'_{δ} 's will make $f(\tau)$ a modular function on $\Gamma_0(n)$. Another interesting special case of this product is given by

$$f_1(\tau) = \prod_{l=1}^g \eta(\delta_l \tau)^{r_{\delta_l}}$$

if we put $\sum_{\delta|n} r'_{\delta} = 0$ in $g(\tau)$.

By imposing different conditions, this time on r_{δ} 's and r'_{δ} 's, we will deduce a transformation law of $f_1(\tau)$.

2.2.1 The Transformation law of $g_1(\tau)$ under $\Gamma_0(n)$

We give a new, detailed proof using residue calculus of the transformation law under $\Gamma_0(n)$.

$$g_1(V\tau) = e^{-\pi i \delta^*} \{-i(c\tau + d)\}^{\frac{1}{2} \sum_{l=1}^g r_{\delta_l} - \frac{1}{2} \sum_{l=1}^g r'_{\delta_l}} g_1(\tau),$$

where

$$g_1(\tau) = \prod_{l=1}^g \frac{\eta(\delta_l \tau)^{r_{\delta_l}}}{\eta(\tau)^{r'_{\delta_l}}}.$$

Consider $V \in \Gamma_0(n)$. Let $a = h'$, $c = k$ and $d = -h$, hence $k = k_l \delta_l$ where $(h, k) = 1$, $k > 0$, $l = 1, 2, \dots, g$ and $hh' \equiv -1 \pmod{k}$. We will write $\tau = (h + iz)/k$ and as a result $V\tau = (h' + iz^{-1})/k$.

We have to prove

$$\begin{aligned}
& - \sum_{l=1}^g \log \eta \left(\frac{\delta_l h' + i \delta_l z^{-1}}{k} \right) r_{\delta_l} + \sum_{l=1}^g \log \eta \left(\frac{\delta_l h + i \delta_l z}{k} \right) r_{\delta_l} + \sum_{l=1}^g \frac{\pi i}{12k_l} (h' - h) r_{\delta_l} \\
& + \sum_{l=1}^g \pi i s(h, k_l) r_{\delta_l} + \log \eta \left(\frac{h' + iz^{-1}}{k} \right) \sum_{l=1}^g r'_{\delta_l} - \log \eta \left(\frac{h + iz}{k} \right) \sum_{l=1}^g r'_{\delta_l} \\
& - \frac{\pi i}{12k} (h' - h) \sum_{l=1}^g r'_{\delta_l} - \pi i s(h, k) \sum_{l=1}^g r'_{\delta_l} \\
& = -\frac{1}{2} \left(\sum_{l=1}^g r_{\delta_l} - \sum_{l=1}^g r'_{\delta_l} \right) \log z.
\end{aligned}$$

The logarithm here is everywhere taken with its principal branch.

Now, from the definition of $\eta(\tau)$,

$$\begin{aligned}
\log \eta \left(\frac{h + iz}{k} \right) &= \frac{\pi i (h + iz)}{12k} + \sum_{m=1}^{\infty} \log(1 - e^{2\pi i h m/k} e^{-2\pi z m/k}) \\
&= \frac{\pi i (h + iz)}{12k} + \sum_{\mu=1}^k \sum_{q=0}^{\infty} \log(1 - e^{2\pi i h \mu/k} e^{-2\pi z (qk + \mu)/k}) \\
&= \frac{\pi i h}{12k} - \frac{\pi z}{12k} - \sum_{\mu=1}^k \sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \frac{1}{r} e^{2\pi i h \mu r/k} e^{-2\pi z (qk + \mu)r/k} \\
&= \frac{\pi i h}{12k} - \frac{\pi z}{12k} - \sum_{\mu=1}^k \sum_{r=1}^{\infty} \frac{1}{r} e^{2\pi i h \mu r/k} \frac{e^{-2\pi z \mu r/k}}{1 - e^{-2\pi z r}}.
\end{aligned}$$

Thus we have to prove that

$$\begin{aligned}
& \sum_{l=1}^g \sum_{\nu=1}^{k_l} \sum_{r=1}^{\infty} \frac{r_{\delta_l}}{r} e^{2\pi i h' \nu r/k_l} \frac{e^{-2\pi \nu r/k_l z}}{1 - e^{-2\pi r/z}} - \sum_{l=1}^g \sum_{\mu=1}^{k_l} \sum_{r=1}^{\infty} \frac{r_{\delta_l}}{r} e^{2\pi i h \mu r/k_l} \frac{e^{-2\pi z \mu r/k_l}}{1 - e^{-2\pi z r}} \\
& + \sum_{l=1}^g \frac{\pi r_{\delta_l}}{12k_l} \left(\frac{1}{z} - z \right) + \pi i \sum_{l=1}^g r_{\delta_l} s(h, k_l) - \sum_{l=1}^g \sum_{\nu=1}^k \sum_{r=1}^{\infty} \frac{r'_{\delta_l}}{r} e^{2\pi i h' \nu r/k} \frac{e^{-2\pi \nu r/k z}}{1 - e^{-2\pi r/z}} \\
& + \sum_{l=1}^g \sum_{\mu=1}^k \sum_{r=1}^{\infty} \frac{r'_{\delta_l}}{r} e^{2\pi i h \mu r/k} \frac{e^{-2\pi z \mu r/k}}{1 - e^{-2\pi z r}} - \sum_{l=1}^g \frac{\pi r'_{\delta_l}}{12k} \left(\frac{1}{z} - z \right) - \pi i s(h, k) \sum_{l=1}^g r'_{\delta_l} \\
& = -\frac{1}{2} \left(\sum_{l=1}^g r_{\delta_l} - \sum_{l=1}^g r'_{\delta_l} \right) \log z.
\end{aligned}$$

We will define a function and calculate the residues of the function at the poles and prove that the sum of the residues is equal to the left side of the above equation. A sort of symmetry is needed between μ and $h\mu$. We introduce therefore

$$\mu^* \equiv h\mu \pmod{k}, \quad (2.7)$$

$$1 \leq \mu^* \leq k - 1.$$

Consider the function

$$\begin{aligned} F_n(x) = & - \frac{1}{4ix} \coth \pi N x \cot \frac{\pi N x}{z} \sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l}) + \sum_{l=1}^g \sum_{\mu=1}^{k_l-1} \frac{r_{\delta_l}}{x} \frac{e^{2\pi\mu N x/k_l}}{1 - e^{2\pi N x}} \frac{e^{-2\pi i \mu^* N x/k_l z}}{1 - e^{-2\pi i N x/z}} \\ & - \sum_{l=1}^g \sum_{\mu=1}^{k_l-1} \frac{r'_{\delta_l}}{x} \frac{e^{2\pi\mu N x/k}}{1 - e^{2\pi N x}} \frac{e^{-2\pi i \mu^* N x/k z}}{1 - e^{-2\pi i N x/z}}, \end{aligned}$$

where $N = n + \frac{1}{2}$. We will integrate $F_n(x)$ along the parallelogram with the vertices $z, i, -z, -i$ and then calculate the residues of this function at its poles and then compare the two answers using the Residue Theorem.

The function $F_n(x)$ has poles at $x = 0$, $x = ir/N$ and $x = -zr/N$ for $r = \pm 1, \pm 2, \pm 3, \dots, \pm n$.

The function

$$- \frac{1}{4ix} \coth \pi N x \cot \frac{\pi N x}{z} \sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})$$

has the residue

$$- \frac{\sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{12i} \left(z - \frac{1}{z} \right)$$

The residue at $x=0$ of

$$\sum_{l=1}^g \frac{r_{\delta_l}}{x} \frac{e^{2\pi\mu N x/k_l}}{1 - e^{2\pi N x}} \frac{e^{-2\pi i \mu^* N x/k_l z}}{1 - e^{-2\pi i N x/z}}$$

is

$$\sum_{l=1}^g \left(\frac{1}{12} - \frac{\mu}{2k_l} + \frac{1}{2} \frac{\mu^2}{k_l^2} \right) r_{\delta_l} z i + \sum_{l=1}^g \left(\frac{\mu}{k_l} - \frac{1}{2} \right) \left(\frac{\mu^*}{k_l} - \frac{1}{2} \right) r_{\delta_l} + \sum_{l=1}^g \left(\frac{1}{12} - \frac{\mu^*}{2k_l} + \frac{1}{2} \frac{\mu^{*2}}{k_l^2} \right) \frac{r_{\delta_l}}{iz}.$$

The sum above has to be summed over μ from 1 to $k_l - 1$. Observe also that μ^* runs from 1 to $k_l - 1$ for all $l = 1, 2, \dots, g$ in the view of (2.7). Also the

first and the third summation are not difficult to calculate. For the middle term, observe from (2.7) that

$$\frac{\mu^*}{k_l} = \frac{h\mu}{k_l} - \left[\frac{h\mu}{k_l} \right],$$

for $l = 1, 2, 3, \dots, g$,

so that

$$\sum_{\mu=1}^{k_l-1} \left(\frac{\mu}{k_l} - \frac{1}{2} \right) \left(\frac{\mu^*}{k_l} - \frac{1}{2} \right) = s(h, k_l).$$

The residue of the remaining function

$$\sum_{l=1}^g \sum_{\mu=1}^{k_l-1} \frac{r'_{\delta_l}}{x} \frac{e^{2\pi\mu Nx/k} e^{-2\pi i \mu^* Nx/kz}}{1 - e^{2\pi Nx} 1 - e^{-2\pi i Nx/z}}$$

is

$$\sum_{l=1}^g \left(\frac{1}{12} - \frac{\mu}{2k} + \frac{1}{2} \frac{\mu^2}{k^2} \right) r'_{\delta_l} z i + \sum_{l=1}^g \left(\frac{\mu}{k} - \frac{1}{2} \right) \left(\frac{\mu^*}{k} - \frac{1}{2} \right) r'_{\delta_l} + \sum_{l=1}^g \left(\frac{1}{12} - \frac{\mu^*}{2k} + \frac{1}{2} \frac{\mu^{*2}}{k^2} \right) \frac{r'_{\delta_l}}{iz}.$$

Thus the residue at $x = 0$ of $F_n(x)$ is

$$\sum_{l=1}^g s(h, k_l) r_{\delta_l} + \sum_{l=1}^g \frac{i r_{\delta_l}}{12k_l} \left(z - \frac{1}{z} \right) - s(h, k) \sum_{l=1}^g r'_{\delta_l} - \sum_{l=1}^g \frac{i r'_{\delta_l}}{12k} \left(z - \frac{1}{z} \right)$$

The residue of $F_n(x)$ at $x = \frac{ir}{N}$ is

$$\frac{\sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{4\pi r} \cot \frac{\pi i r}{z} - \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu=1}^{k_l-1} \frac{r_{\delta_l}}{r} e^{2\pi i \mu r/k_l} \frac{e^{2\pi \mu^* r/k_l z}}{1 - e^{2\pi r/z}} + \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu=1}^{k-1} \frac{r'_{\delta_l}}{r} e^{2\pi i \mu r/k} \frac{e^{2\pi \mu^* r/kz}}{1 - e^{2\pi r/z}}.$$

It is easy to see that

$$h'\mu \equiv hh'\mu \equiv -\mu \pmod{k_l},$$

for $l = 1, 2, 3, \dots, g$,

and

$$h'\mu \equiv hh'\mu \equiv -\mu \pmod{k}.$$

As a result we get

$$\begin{aligned} \frac{\sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{4\pi i r} \coth \frac{\pi r}{z} &= \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu=1}^{k_l-1} \frac{r_{\delta_l}}{r} e^{-2\pi i h' \mu^* r / k_l} \frac{e^{2\pi \mu^* r / k_l z}}{1 - e^{2\pi r / z}} \\ &+ \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu=1}^{k-1} \frac{r'_{\delta_l}}{r} e^{-2\pi i h' \mu^* r / k} \frac{e^{2\pi \mu^* r / k z}}{1 - e^{2\pi r / z}}. \end{aligned}$$

The parallelogram contains the poles $x = \frac{ir}{N}$ for $-n \leq r \leq -1$ and $1 \leq r \leq n$.

We sum then over the poles and we get

$$\begin{aligned} &\frac{\sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{2\pi i} \sum_{r=1}^n \frac{1}{r} \left(\frac{2e^{-2\pi r / z}}{1 - e^{-2\pi r / z}} + 1 \right) + \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu^*=1}^{k_l-1} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h' \mu^* r / k_l} \frac{e^{-2\pi \mu^* r / k_l z}}{1 - e^{-2\pi r / z}} \\ &- \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu^*=1}^{k_l-1} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h' (k_l - \mu^*) r / k_l} \frac{e^{-2\pi (k_l - \mu^*) r / k_l z}}{e^{-2\pi r / z} - 1} \\ &- \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu^*=1}^{k-1} \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h' \mu^* r / k} \frac{e^{-2\pi \mu^* r / k z}}{1 - e^{-2\pi r / z}} \\ &+ \frac{1}{2\pi i} \sum_{l=1}^g \sum_{\mu^*=1}^{k-1} \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h' (k - \mu^*) r / k} \frac{e^{-2\pi (k - \mu^*) r / k z}}{e^{-2\pi r / z} - 1}. \end{aligned}$$

In the third and fifth sum we replace $k_l - \mu^*$ and $k - \mu^*$ by μ^* and combine it with the other sum. As a result the residue of $F_n(x)$ at $x = \frac{ir}{N}$ is given by

$$\begin{aligned} &\frac{\sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{2\pi i} \sum_{r=1}^n \frac{1}{r} + \frac{1}{\pi i} \sum_{l=1}^g \sum_{\nu=1}^{k_l} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h' \nu r / k_l} \frac{e^{-2\pi \nu r / k_l z}}{1 - e^{-2\pi r / z}} \\ &- \frac{1}{\pi i} \sum_{l=1}^g \sum_{\nu=1}^k \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h' \nu r / k} \frac{e^{-2\pi \nu r / k z}}{1 - e^{-2\pi r / z}}. \end{aligned}$$

Similarly, we find the sum of the residues of $F_n(x)$ at $x = -\frac{zr}{N}$, $r = \pm 1, \pm 2, \pm 3, \dots, \pm n$ is given by

$$\begin{aligned} &\frac{i \sum_{l=1}^g (r_{\delta_l} - r'_{\delta_l})}{2\pi} \sum_{r=1}^n \frac{1}{r} + \frac{i}{\pi} \sum_{l=1}^g \sum_{\nu=1}^{k_l} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h \nu r / k_l} \frac{e^{-2\pi \nu r z / k_l}}{1 - e^{-2\pi r z}} \\ &- \frac{i}{\pi} \sum_{l=1}^g \sum_{\nu=1}^k \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h \nu r / k} \frac{e^{-2\pi \nu r z / k}}{1 - e^{-2\pi r z}}. \end{aligned}$$

Thus the sum of all the residues of $F_n(x)$ within the parallelogram is

$$\begin{aligned} & \sum_{l=1}^g \frac{r_{\delta_l}}{12k_l i} \left(\frac{1}{z} - z \right) + \sum_{l=1}^g s(h, k_l) r_{\delta_l} + \frac{1}{\pi i} \sum_{l=1}^g \sum_{\nu=1}^{k_l} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h' \nu r / k_l} \frac{e^{-2\pi \nu r / k_l z}}{1 - e^{-2\pi r / z}} \\ & - \frac{1}{\pi i} \sum_{l=1}^g \sum_{\mu=1}^{k_l} \sum_{r=1}^n \frac{r_{\delta_l}}{r} e^{2\pi i h \mu r / k_l} \frac{e^{-2\pi \mu r z / k_l}}{1 - e^{-2\pi r z}} - \sum_{l=1}^g \frac{r'_{\delta_l}}{12k_l i} \left(\frac{1}{z} - z \right) - \sum_{l=1}^g s(h, k) r'_{\delta_l} \\ & - \frac{1}{\pi i} \sum_{l=1}^g \sum_{\nu=1}^k \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h' \nu r / k} \frac{e^{-2\pi \nu r / k z}}{1 - e^{-2\pi r / z}} + \frac{1}{\pi i} \sum_{l=1}^g \sum_{\mu=1}^k \sum_{r=1}^n \frac{r'_{\delta_l}}{r} e^{2\pi i h \mu r / k} \frac{e^{-2\pi \mu r z / k}}{1 - e^{-2\pi r z}}. \end{aligned}$$

What remains to prove is that

$$\lim_{n \rightarrow \infty} \int_C F_n(x) dx = - \left(\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l} \right) \log z,$$

where C is the parallelogram of vertices $z, i, -z, -i$.

Now on the four sides of C , except at the vertices, the second and the third summands in $F_n(x)$ goes to zero as n goes to infinity. Now regarding the first part of the function, it is easy to see that

$$\lim_{n \rightarrow \infty} \coth \pi N x \cot \frac{\pi N x}{z} = i$$

on the sides i to $-z$ and $-i$ to z and that

$$\lim_{n \rightarrow \infty} \coth \pi N x \cot \frac{\pi N x}{z} = -i$$

on the sides i to z and $-i$ to $-z$.

Therefore

$$\lim_{n \rightarrow \infty} F_n(x) = \frac{\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l}}{4}$$

on the sides i to z and on $-i$ to $-z$, and

$$\lim_{n \rightarrow \infty} F_n(x) = - \frac{\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l}}{4}$$

on the sides i to $-z$ and on $-i$ to z . The convergence of $F_n(x)$ is not uniform but it is bounded since the denominators of the three summands are bounded

away from zero and this is because $N = n + \frac{1}{2}$ is not an integer. We then have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_C F_n(x) dx &= \frac{\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l}}{4} \left\{ -\int_{-i}^z \frac{dx}{x} + \int_z^i \frac{dx}{x} - \int_i^{-z} \frac{dx}{x} + \int_{-z}^{-i} \frac{dx}{x} \right\} \\
&= \frac{\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l}}{2} \left\{ -\int_{-i}^z \frac{dx}{x} + \int_z^i \frac{dx}{x} \right\} \\
&= \frac{\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l}}{2} \left\{ -\left(\log z + \frac{\pi i}{2} \right) + \left(\frac{\pi i}{2} - \log z \right) \right\} \\
&= -\left(\sum_{l=1}^g r_{\delta_l} - r'_{\delta_l} \right) \log z.
\end{aligned}$$

2.2.2 A Special Case of $g_1(\tau)$

Let

$$f(\tau) = \prod_{l=1}^g \left(\frac{\eta(\delta_l \tau)}{\eta(\tau)} \right)^{r_{\delta_l}}.$$

Also suppose that

$$\frac{1}{24} \sum_{l=1}^g (\delta_l - 1) r_{\delta_l} \tag{2.8}$$

is an integer and

$$\frac{1}{24} \sum_{l=1}^g (\delta'_l - n) r_{\delta_l} \tag{2.9}$$

is an integer,

where $n = \delta_l \delta'_l$,

$$\prod_{l=1}^g \delta_l^{r_{\delta_l}} \tag{2.10}$$

is a rational square, and $r_1 = 0$.

It is easy to see that $f(\tau)$ is the special case of $g_1(\tau)$ in which $r_\delta = r'_\delta$ for all δ dividing n . We then have

$$f(V\tau) = e^{-\pi i \delta^{**}} f(\tau),$$

where

$$\delta^{**} = \sum_{l=1}^g \left\{ \left\{ \frac{a+d}{12c} + s(-d, c) \right\} - \left\{ \frac{a+d}{12c_l} + s(-d, c_l) \right\} \right\} r_{\delta_l}.$$

Suppose now that $(a, 6) = 1$ and $c > 0$. M. Newman [15] using (2.8), (2.9) and (2.10) showed that

$$\sum_{l=1}^g \left\{ \left\{ \frac{a+d}{12c} + s(-d, c) \right\} - \left\{ \frac{a+d}{12c_l} + s(-d, c_l) \right\} \right\} r_{\delta_l}$$

is an even integer. Hence,

$$f(V\tau) = f(\tau)$$

where $V \in \Gamma_0(n)$.

In [15], M. Newman mentioned that since $S = \tau + 1$ is in $\Gamma_0(n)$ for every n , $\Gamma_0(n)$ can be generated by the elements

$$\begin{pmatrix} a & b \\ nc & d \end{pmatrix},$$

where $(a, 6) = 1$. Thus it is necessary to show the invariance of a function only with respect to these transformations in order to show its invariance for $\Gamma_0(n)$. Also, it suffices to consider only these substitutions for which both a and nc are positive.

2.2.3 Another Special Case of $g_1(\tau)$

Let

$$f_1(\tau) = \prod_{l=1}^g \eta(\delta_l \tau)^{r_{\delta_l}},$$

where

$$\sum_{l=1}^g \delta_l r_{\delta_l} \equiv 0 \pmod{24} \tag{2.11}$$

and

$$\sum_{l=1}^g \frac{n}{\delta_l} r_{\delta_l} \equiv 0 \pmod{24}. \tag{2.12}$$

Let $k = \frac{1}{2} \sum_{l=1}^g r_{\delta_l} \in \mathbb{Z}$. It is easy to see that $f_1(\tau)$ is a special case of $g_1(\tau)$ where $\sum_{l=1}^g r'_{\delta_l} = 0$. We then have

$$f_1(V\tau) = e^{-\pi i \delta^{***}} \{-i(c\tau + d)\}^k f_1(\tau),$$

where

$$\delta^{***} = \sum_{l=1}^g \left\{ -\frac{a+d}{12c_l} - s(-d, c_l) \right\} r_{\delta_l}.$$

We have to prove now that the transformation law above is the same as

$$f_1(V\tau) = \chi(d)(c\tau + d)^k f_1(\tau),$$

where $V \in \Gamma_0(n)$ and

$$\chi(d) = \left(\frac{(-1)^k \prod_{l=1}^g \delta_l^{r_{\delta_l}}}{d} \right).$$

Since k is an integer, we get

$$f_1(V\tau) = e^{-\pi i \delta^{***}} (-i)^k (c\tau + d)^k f_1(\tau)$$

What remains to prove is that

$$\chi(d) = (-i)^k e^{-\pi i \delta^{***}}.$$

Notice that $-ad \equiv -1 \pmod{c}$. Thus $s(-d, c) = -s(a, c)$.

We have that

$$\delta M\tau = \delta \begin{pmatrix} a & b \\ nc_1 & d \end{pmatrix} \tau = \begin{pmatrix} a & \delta b \\ \delta' c_1 & d \end{pmatrix} \delta\tau = M_1\delta\tau,$$

where $M \in \Gamma_0(n)$.

Thus $\eta(\delta M\tau) = \eta(M_1\delta\tau)$ and so

$$f(M\tau) = \prod_{l=1}^g \eta(\delta_l M\tau)^{r_{\delta_l}} = \prod_{l=1}^g \eta(M_1\delta_l\tau)^{r_{\delta_l}}.$$

Assume now that $(a, 6) = 1$, $c > 0$ and $n = \delta_l \delta'_l$. In [15], Newman proved that

$$s(a, c) - (a+d)/12c \equiv \frac{1}{12}a(c-b-3) - \frac{1}{2} \left\{ 1 - \left(\frac{c}{a} \right) \right\} \pmod{2},$$

where $\left(\frac{c}{a}\right)$ is the generalized Legendre-Jacobi symbol of the quadratic reciprocity. Write $c = c_1 n$. Thus

$$\begin{aligned}\delta^{***} &= \sum_{l=1}^g \left\{ s(a, \delta'_l c_1) - \frac{(a+d)}{12\delta'_l c_1} \right\} r_{\delta_l} \\ &\equiv \frac{ac_1}{12} \sum_{l=1}^g \delta'_l r_{\delta_l} - \frac{ab}{12} \sum_{l=1}^g \delta_l r_{\delta_l} - \frac{3a}{12} \sum_{l=1}^g r_{\delta_l} - \frac{1}{2} \sum_{l=1}^g \left\{ 1 - \left(\frac{\delta'_l c_1}{a} \right) \right\} r_{\delta_l} \pmod{2} \\ &\equiv -\frac{2ac_1}{24} \sum_{l=1}^g \delta'_l r_{\delta_l} + \frac{2ab}{24} \sum_{l=1}^g \delta_l r_{\delta_l} + \frac{k}{2} + \frac{1}{2} \sum_{l=1}^g \left\{ 1 - \left(\frac{\delta_l}{a} \right) \right\} r_{\delta_l} \pmod{2}.\end{aligned}$$

But we are given that $\sum_{l=1}^g \delta'_l r_{\delta_l} \equiv 0 \pmod{24}$ and $\sum_{l=1}^g \delta_l r_{\delta_l} \equiv 0 \pmod{24}$.

Thus

$$\frac{ac_1}{12} \sum_{l=1}^g \delta'_l r_{\delta_l}$$

and

$$\frac{ab}{12} \sum_{l=1}^g \delta_l r_{\delta_l}$$

are even integers. Therefore, we get

$$\begin{aligned}e^{-\pi i \delta^{***}} &= e^{\pi i \frac{1}{2} k} e^{\pi i \frac{1}{2} \sum_{l=1}^g \left\{ 1 - \left(\frac{\delta_l}{a} \right) \right\} r_{\delta_l}} \\ &= (-i)^k e^{\pi i \frac{1}{2} \sum_{l=1}^g \left\{ 1 - \left(\frac{\delta_l}{a} \right) \right\} r_{\delta_l}}.\end{aligned}$$

Now,

$$e^{\pi i \frac{1}{2} \sum_{l=1}^g \left\{ 1 - \left(\frac{\delta_l}{a} \right) \right\} r_{\delta_l}} = \prod_{l=1}^g \left(\frac{\delta_l}{a} \right)^{r_{\delta_l}}.$$

Thus

$$(-i)^k e^{-\pi i \delta^{***}} = \left(\frac{(-1)^k \prod_{l=1}^g \delta_l^{r_{\delta_l}}}{a} \right).$$

But $ad - bc = 1$, as a result we get

$$(-i)^k e^{-\pi i \delta^{***}} = \left(\frac{(-1)^k \prod_{l=1}^g \delta_l^{r_{\delta_l}}}{d} \right),$$

and hence

$$f_1(V\tau) = \chi(d)(c\tau + d)^k f_1(\tau)$$

where $\chi(d) = \left(\frac{(-1)^k \prod_{l=1}^g \delta_l^{r_{\delta_l}}}{d} \right)$.

2.3 Comments on Generalizing the Proof of Section (2.1)

In this section, we define $\theta_3(w, \tau)$ in several variables. We call it $G(w, \tau)$. We give several steps in a process that allows us to generalize the proof of the transformation law of $\theta_3(w, \tau)$ to $G(w, \tau)$. So we let $w = (w_1, w_2, w_3, \dots, w_s) \in \mathbb{C}$ be a complex s-tuple. The function $G(w, \tau)$ is defined by,

$$G(w, \tau) = \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{r=1}^s \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2w_r + q^{4n-2}) , \quad (2.13)$$

where $q = e^{\pi i \tau}$ and τ is in the upper half plane. The transformation law is given by,

$$G\left(\frac{w}{\tau}, \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{-\frac{|w|^2}{\pi i \tau}} e^{\frac{(1-s)\pi i(\tau+(1/\tau))}{12}} G(w, \tau) . \quad (2.14)$$

First, we prove (2.14) for $\tau = iy$, where $y > \frac{2|t|}{\pi}$, where $w = \sigma + it$, then extend the result to all τ in the upper half plane by analytic continuation. In proving the transformation law of the logarithmic derivative, we will encounter some problems with the zeroes of the theta function. The zeroes of $G(w, \tau)$ are at $w_r = \frac{\pi}{2} + \frac{\pi\tau}{2} + m\pi + n\pi\tau$, for $m, n \in \mathbb{Z}$ for any r .

To solve this problem, we first fix w_r such that $\text{Re} w_r \neq \frac{\pi}{2} + n\pi$ for every r and prove the transformation law for $\tau = iy$ and prove (2.14) using the logarithmic derivative. We then deduce (2.14) and extend the result by analytic continuation to the whole τ plane. Once we have it for all w such that $\text{Re} w_r \neq \frac{\pi}{2} + n\pi$ for all r we use analytic continuation in the w space to extend the result to all w .

Theorem 2.2 *If $\tau = iy$ and $y > \frac{2|t|}{\pi}$, where $w = (\sigma_1 + it_1, \sigma_2 + it_2, \dots, \sigma_s + it_s)$ and $|t| = (\sum_{r=1}^s t_r^2)^{\frac{1}{2}}$, the transformation formula is*

$$G\left(\frac{w}{iy}, \frac{i}{y}\right) = (y)^{\frac{1}{2}} e^{-(1-s)\pi(y-1/y)} e^{\frac{|w|^2}{\pi y}} G(w, iy) . \quad (2.15)$$

So we follow the same steps as in the proof of $\theta_3(w, \tau)$, by expanding (2.15) in terms of its Taylor series. As before, we will use residue calculus to prove the transformation law. To prove this, consider

$$F_n(z) = -\frac{1}{8z} \cot \pi i N z \cot \pi N z / y + \frac{1}{z} \sum_{r=1}^s \left(\frac{e^{-iNz(\pi/y + 2w_r/y + 2w_r)}}{1 - e^{-2\pi i N z / y}} \right) \left(\frac{e^{Nz(\pi + 2iw_r)}}{1 - e^{2\pi z N}} \right),$$

where $N = n + \frac{1}{2}$.

We then calculate the residues of $F_n(z)$ at the poles $z = 0$, $z = \frac{ik}{N}$ and $z = \frac{ky}{N}$ for $k = \pm 1, \pm 2, \dots$.

So we get

$$\begin{aligned} 2\pi i \sum_{\substack{k=-n \\ z=\frac{ky}{N}; z=\frac{ik}{N}}}^n \text{Res} F_n(z) &= \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{1 - e^{2\pi y k}} \right) + \sum_{r=1}^s \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2iw_r k} \left(\frac{e^{\pi y k}}{1 - e^{2\pi y k}} \right) \\ &+ \sum_{r=1}^s \sum_{k=1}^n \frac{(-1)^k}{k} e^{2iw_r k} \left(\frac{e^{\pi y k}}{1 - e^{2\pi y k}} \right) - \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{1 - e^{2\pi k/y}} \right) \\ &- \sum_{r=1}^s \sum_{k=1}^n \frac{(-1)^k}{k} e^{-2kw_r/y} \left(\frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \right) \\ &- \sum_{r=1}^s \sum_{k=1}^n \frac{(-1)^k}{k} e^{2w_r k/y} \left(\frac{e^{\pi k/y}}{1 - e^{2\pi k/y}} \right) + \frac{|w|^2}{\pi y} \\ &- \frac{(1-s)\pi(y - (1/y))}{12}. \end{aligned}$$

Note also that

$$\lim_{n \rightarrow \infty} \oint_C F_n(z) dz = -\frac{1}{2} \log y,$$

where C is the parallelogram of vertices $y, i, -y$ and $-i$ taken counterclockwise.

As a result, we get Theorem 2.2 using residue calculus.

CHAPTER 3

Generalized Modular Forms Representable As Eta-Products

3.1 Introduction

Let τ be in the upper half plane and $n \in \mathbb{Z}$. The Dedekind eta function is defined by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

and the Generalized Dedekind η -function is defined by

$$\eta_{\delta, g}(\tau) = e^{\pi i P_2(\frac{g}{\delta}) \delta \tau} \prod_{\substack{m > 0 \\ m \equiv g \pmod{\delta}}} (1 - x^m) \prod_{\substack{m > 0 \\ m \equiv -g \pmod{\delta}}} (1 - x^m),$$

where $x = e^{2\pi i \tau}$, $\tau \in H$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} = t - [t]$ is the fractional part of t . Note that $\eta_{\delta, 0}(\tau) = \eta(\delta\tau)^2$ and that $\eta_{\delta, \delta/2}(\tau) = \frac{\eta^2((\delta/2)\tau)}{\eta^2(\delta\tau)}$.

Consider

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, c \equiv 0 \pmod{N}, ad - bc = 1 \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, c \equiv 0 \pmod{N}, a \equiv d \equiv 1 \pmod{N}, ad - bc = 1 \right\},$$

and

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N}, ad - bc = 1 \right\},$$

which are congruence subgroups of the full modular group.

Definition 3.1 *A generalized modular form (GMF) of weight k on Γ is a function $f(\tau)$ meromorphic throughout the complex upper plane H , which is also meromorphic at the cusps and satisfies the transformation law*

$$f(M\tau) = \nu(M)(c\tau + d)^k f(\tau),$$

for all $M \in \Gamma$. Here we allow the possibility that $|\nu(M)| \neq 1$.

In [9], Kohnen and Mason presented the proof of the following theorem.

Theorem 3.1 *Let f be a GMF of weight 0. Assume that f has no poles or zeroes in $H \cup \mathbb{Q} \cup \infty$. Assume furthermore that Γ is a congruence subgroup and that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Then f is constant.*

Afterwards, they considered the subgroup $\Gamma_0(N)$ and proved that a GMF with its zeroes and poles supported at the cusps, and such that the order of the function at the cusp is independent of the numerator of that cusp with the above conditions on the Fourier coefficients, is a classical eta-product. Their result is given in Theorem 3.2 below. In this chapter we replace the condition imposed by Kohnen and Mason on the order of the function at the cusp by a condition on N . We then prove a theorem with conditions at the cusps which are similar to those of Kohnen and Mason, but on $\Gamma_1(N)$ instead of $\Gamma_0(N)$.

It will turn out that functions with such conditions upon the order of the function at the cusps are also representable as eta products on $\Gamma_1(N)$. Finally, we deduce a similar theorem on $\Gamma(N)$.

The theorem of Kohnen and Mason on the subgroup $\Gamma_0(N)$ is as follows. Note that a complete set of representatives of the cusps of $\Gamma_0(N)$ [9] is given by

$$\frac{a}{c} \tag{3.1}$$

where c divides N and a is taken modulo N , with $(a, N) = 1$ and the a 's are inequivalent modulo $(c, \frac{N}{c})$.

Theorem 3.2 *Let f be a GMF of integral weight k on $\Gamma_0(N)$. Suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Suppose further that the order of the function f at each cusp of $\Gamma_0(N)$ is independent of a . Then f is an eta-quotient, i.e. there are integers $M \neq 0$ and $m_t(t \mid N)$ such that*

$$f^M(\tau) = c \prod_{t \mid N} \Delta(t\tau)^{m_t}.$$

where $\Delta(\tau) = \eta(\tau)^{24}$.

Notice that $f^M / \prod_{t \mid N} \Delta(t\tau)^{m_t}$ has Fourier coefficients at $i\infty$ which are rational and p integral. This is due to the fact that the product in the denominator has integer coefficients with 1 as a leading coefficient. Theorem 3.2 then easily follows from Theorem 3.1.

We can now present special cases of this theorem. If $N = p$, where p is a prime, we have two cusps for $\Gamma_0(p)$, one with denominator 1 and one with denominator p , so the condition on the order of the function at the cusps is automatic. For N square free, this condition is automatic, since for each divisor of N , we have only one cusp whose denominator is this divisor. To modify the condition at the order of the cusps we define a class of functions which is a form on $\Gamma_1(N)$ and then lift it by applying a coset operator.

We now present another class of functions called the Generalized Dedekind η -Products. Consider

$$f(\tau) = \prod_{\substack{\delta|N \\ g}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \quad (3.2)$$

where $0 \leq g < \delta$ and $r_{\delta,g}$ are integers and may be half integers only if $g = 0$ or $g = \frac{\delta}{2}$ (we allow half integers in order to include the ordinary eta products).

In [20], S. Robins proved that (3.2) is a modular function on $\Gamma_1(N)$ by imposing certain conditions on the $r_{\delta,g}$'s. It will be sufficient for our purposes to note that the above function is a classical modular form on $\Gamma_1(N)$ with a multiplier system. For $A \in \Gamma_1(N)$, the transformation law of the $f(\tau)$ is given by

$$f(A\tau) = f(\tau)e^{\pi i \sum \mu_{\delta,g} r_{\delta,g}},$$

where

$$\mu_{\delta,g} = \frac{\delta a}{c} P_2\left(\frac{g}{\delta}\right) + \frac{\delta d}{c} P_2\left(\frac{ag}{\delta}\right) - 2s\left(a, \frac{c}{\delta}, 0, \frac{g}{\delta}\right)$$

and $s(h, k, x, y)$ is the Meyer sum, a generalized Dedekind Sum, defined by

$$s(h, k, x, y) = \sum_{\mu \bmod k} \left(\left(h \left(\frac{\mu + y}{k} \right) + x \right) \right) \left(\left(\frac{\mu + y}{k} \right) \right).$$

As usual $((x)) = x - [x] - \frac{1}{2}$ if x is not an integer and 0 otherwise.

3.2 GMF's on $\Gamma_0(N)$ Representable as Generalized Eta-Products

As we have already pointed out, a complete set of representatives of the cusps of $\Gamma_0(N)$ is given by

$$\frac{a}{c} \quad (3.3)$$

where c is a positive divisor of N and a runs through integers with $1 \leq a \leq N$, $(a, N) = 1$ that are inequivalent modulo $(c, \frac{N}{c})$. The width of the cusp $\frac{a}{c}$ in (3.3) is given by

$$w_{a/c} = \frac{N}{(c^2, N)}.$$

Theorem 3.3 *Let f be a GMF of rational weight k' on $\Gamma_0(N)$. Suppose that the poles and zeroes of f are supported at the cusps. Suppose further that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p and that the rank of*

$$((\delta, c)^2 P_2(ag/(\delta, c)))_{(\delta|N, 0 \leq g < \delta), (c|N, a)} \quad (3.4)$$

is equal to the number of cusps, where the columns of the matrix corresponds to the cusps a/c of $\Gamma_0(N)$. Then f is a classical modular form.

Remark: Note that the rank of the above matrix is less than or equal to the number of cusps.

Proof

For given integers $r_{\delta, g}$ put

$$F(\tau) = \prod_{\delta|N} \prod_{0 \leq g < \delta} \eta_{\delta, g}(\tau)^{r_{\delta, g}},$$

F is a modular form on $\Gamma_1(N)$ of weight $k = \sum r_{\delta, 0}$ and by [20],

$$\text{ord}_{a/c} F = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0 \leq g < \delta} \frac{(\delta, c)^2}{\delta} P_2 \left(\frac{ag}{(\delta, c)} \right) r_{\delta, g}.$$

We now consider the cosets of $\Gamma_1(N)$ in $\Gamma_0(N)$. By applying the operator defined below, we lift the above generalized eta product from a modular form on $\Gamma_1(N)$ to a modular form on $\Gamma_0(N)$. For $\beta_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and F a function on H , we define the following operator

$$F |_k \beta_j = (c_j \tau + d_j)^{-k} F(\beta_j \tau).$$

Let

$$H(F) = \prod_j F |_k \beta_j$$

where $\{\beta_j\}$ are coset representatives. We see that

$$F(\tau) |_k \beta_j = (c_j \tau + d_j)^{-k} \prod_{\delta|N} \prod_{0 \leq g < \delta} \eta_{\delta, g}(\beta_j \tau)^{r_{\delta, g}}.$$

Recall that F is a modular form on $\Gamma_1(N)$ of weight k . It follows that $H(F)$ is a modular form on the larger group $\Gamma_0(N)$ of weight $k_1 = |\Gamma_1(N) \backslash \Gamma_0(N)| k$. We have to determine the order of $H(F)$ at any cusp of $\Gamma_0(N)$. We have to show first that after we apply the operator we again get an eta product and that the operator will not affect the order of the function at the cusps as calculated in [20]. Recall that Robins presented the transformation of $\eta_{\delta,g}$ under $A \in \Gamma_0(N)$.

For $g \neq 0$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have,

$$\eta_{\delta,g}(A\tau) = e^{\pi i \mu_{\delta,g}} \eta_{\delta,ag}(\tau) \quad (3.5)$$

where

$$\mu_{\delta,g} = \frac{\delta a}{c} P_2\left(\frac{g}{\delta}\right) + \frac{\delta d}{c} P_2\left(\frac{ag}{\delta}\right) - 2s\left(a, \frac{c}{\delta}, 0, \frac{g}{\delta}\right)$$

and $s(h, k, x, y)$ is the Meyer Sum, a generalized Dedekind sum.

Thus if $\beta_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and for a given δ where $(a_j, \delta) = 1$, if $0 \leq g < \delta$, then $a_j g$ will run through a complete set of representatives modulo δ and also for a given δ , if $g_1 \equiv -g_2 \pmod{\delta}$, then

$$\prod_{\substack{m>0 \\ m \equiv g_1 \pmod{\delta}}} (1-x^m) \prod_{\substack{m>0 \\ m \equiv -g_1 \pmod{\delta}}} (1-x^m) = \prod_{\substack{m>0 \\ m \equiv g_2 \pmod{\delta}}} (1-x^m) \prod_{\substack{m>0 \\ m \equiv -g_2 \pmod{\delta}}} (1-x^m)$$

and

$$P_2\left(\frac{g_1}{\delta}\right) = P_2\left(\frac{k\delta - g_2}{\delta}\right) = P_2\left(1 - \frac{g_2}{\delta}\right) = P_2\left(\frac{g_2}{\delta}\right),$$

by (3.5). Hence

$$\eta_{\delta,g_1} = \eta_{\delta,g_2}.$$

Recall also that $\beta_j \in \Gamma_0(N)$. Also $\eta_{\delta,0}$ and $\eta_{\delta,\delta/2}$ are forms on $\Gamma_0(\delta)$ and hence on $\Gamma_0(N)$. As a result, we have

$$\begin{aligned} F(\tau) |_k \beta_j &= \nu_j (c_j \tau + d_j)^{-k} \prod_{\delta|N} \prod_{0 \leq g < \delta} \eta_{\delta,g}(\beta_j \tau)^{r_{\delta,g}} \\ &= \nu_j (c_j \tau + d_j)^{-k} \prod_{\delta|N} \eta_{\delta,0}(\beta_j \tau)^{r_{\delta,0}} \prod_{\delta|N} \prod_{0 < g < \delta} \eta_{\delta,g}(\beta_j \tau)^{r_{\delta,g}} \\ &= \nu_j^* \prod_{\delta|N} \eta_{\delta,0}(\tau)^{r_{\delta,0}} \prod_{\delta|N} \prod_{0 < g < \delta} \eta_{\delta,g}(\tau)^{r_{\delta,g}}, \end{aligned}$$

where the exponents $r_{\delta,g}$ are renamed according to the new values of g and ν_j is a constant depending on β_j . Thus we have

$$H(F) = \nu \prod_{\delta|N} \prod_{0 \leq g < \delta} \eta_{\delta,g}(\tau)^{r'_{\delta,g}}$$

where again $r'_{\delta,g} = \sum_j r_{\delta,a_j g}$ are new exponents and ν is a constant depending on β_j for all j . Using the condition (3.4), we have to solve now for $r'_{\delta,g}$. To do this we have to determine the order of $H(F)$ at the cusps. In [20], it is given that

$$\text{ord}_{a/c} \prod_{\delta|N} \eta_{\delta,g}^{r_{\delta,g}} = \frac{w_{a/c}}{2} \sum_{\delta|N} \frac{(\delta, c)^2}{\delta} P_2 \left(\frac{ag}{(\delta, c)} \right) r_{\delta,g}.$$

Thus

$$\text{ord}_{a/c} H(F) = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0 \leq g < \delta} \frac{(\delta, c)^2}{\delta} P_2 \left(\frac{ag}{(\delta, c)} \right) r'_{\delta,g}.$$

Notice now that the product of two expressions whose Fourier coefficients are rational and p-integral for all but a finite number of primes has rational Fourier coefficients that are p-integral for all but a finite number of primes p. $\eta_{\delta,g}(\tau)$ has Fourier coefficients that are rational and p-integral for all but a finite number of primes. Thus the Fourier coefficients of $H(F)$ are rational and p-integral for all but a finite number of primes since $H(F)$ turned to be a generalized eta product. We still want to show that $r'_{\delta,g}$ can be chosen so that

$$\text{ord}_{a/c} H(F) = m h_{a/c}, \tag{3.6}$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f . By hypothesis, the rank of

$$\left((\delta, c)^2 P_2(ag/(\delta, c)) \right)_{(\delta|N, 0 \leq g < \delta), (c|N, a)}$$

is equal to the number of cusps. Therefore we can choose $r'_{\delta,g}$ so that (3.6) is satisfied, with an appropriate m . Since η does not vanish on H , we find from

the valence formula applied to $H(F)$ that the sum of the orders of $H(F)$ at the different cusps of $\Gamma_0(N)$ is equal to

$$\frac{k_1}{12}[\Gamma(1) : \Gamma_0(N)].$$

On the other hand, the valence formula is also valid for the GMF f of weight k' [8]. We then deduce from (3.6) that

$$k_1 = mk'.$$

We see that $f^m/H(F)$ is a GMF satisfying all the assumptions of Theorem 3.1. We conclude that $f^m = cH(F)$, as required.

Note that for $N = p_1p_2\dots p_n$ square-free, the cusps of $\Gamma_0(N)$ are $1/1, 1/p_1, 1/p_2, \dots, 1/p_n$ and $1/N$. Hence $\Gamma_0(N)$ satisfies the condition at the order of the cusps given in the paper of Kohnen and Mason.

For $N = p^2$, the condition of Kohnen and Mason fails and the condition of Theorem 3.3 fails too. This happens because $P_2(a_1g/(\delta, c)) = P_2(a_2g/(\delta, c))$ for $c = p$ and for all $a_1 \equiv -a_2 \pmod{p}$ for all g and thus

$$\left((\delta, c)^2 P_2(ag(\delta, c)) \right)_{(\delta|N, 0 \leq g < \delta), (c|N, a)}$$

has a rank smaller than the number of cusps.

3.3 GMF's on $\Gamma_1(N)$ Representable as Eta-Products

Every cusp of $\Gamma_1(N)$ is equivalent to

$$\frac{a}{c} \tag{3.7}$$

where c is taken modulo N and a is taken modulo $d = (N, c)$ and $(a, d) = 1$. Moreover, for every cusp of $\Gamma_1(N)$ there exist precisely two fractions a/c of the above form that are equivalent to that cusp. The width of the every cusp in (3.7) is given by

$$w_{a/c} = \frac{N}{(c, N)}.$$

Theorem 3.4 *Let f be a GMF of integral weight k on $\Gamma_1(N)$. Suppose that the poles and zeroes of f are supported at the cusps and that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Suppose further that the order of the function f at each cusp of $\Gamma_1(N)$ is independent of a and for the cusps a_1/c_1 whose denominator does not divide N , the function will have the same order at a_1/c_1 as at those cusps whose denominators are (c_1, N) . Then f is an eta quotient, i.e., there are integers $M \neq 0$ and $m_t (t \mid N)$ such that*

$$f^M(\tau) = c \prod_{t \mid N} \Delta(t\tau)^{m_t}.$$

Proof We have

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

For given integers m_t put

$$F(\tau) = \prod_{t \mid N} \Delta(t\tau)^{m_t}.$$

Then F is a modular form on $\Gamma_1(N)$ and

$$\text{ord}_{a/c} F = w_{a/c} \left(\sum_{t \mid N} \frac{(t, c)^2}{t} m_t \right).$$

Note that the order at every cusp a/c is independent of a and hence F itself satisfies the order condition given by Theorem 3.4. Moreover the conditions imposed in the theorem are important since $\sum_{t \mid N}$ in the above expression for $\text{ord}_{a/c}$ runs only over the divisors of N . We want to show that m_t can be chosen so that

$$\text{ord}_{a/c} F = m h_{a/c} \tag{3.8}$$

for all cusps a/c of $\Gamma_1(N)$. Here $h_{a/c}$ is the order of f^{12} at a/c and m is an appropriate non-zero integer depending only on f . Note that by assumption $h_{a/c}$ is independent of a . Note that in our case for $\Gamma_1(N)$, the denominator of the cusp is taken modulo N , not as a divisor of N as in the case of $\Gamma_0(N)$. Since

we are given that the order of the function at the cusp a/c whose denominator doesn't divide N is equal to the order of the function at the cusp whose denominator is (c, N) , there are $\sigma_0(N)$ equations. So as in the proof of Mason and Kohnen, it will be sufficient to prove that the square matrix

$$A_N = ((t, c)^2)_{t|N, c|N}$$

of size $\sigma_0(N) \times \sigma_0(N)$ is invertible. Now using [2], we see that

$$A'_N = ((t, c))_{t|N, c|N}$$

is positive definite and hence invertible. The Oppenheim Inequality [9] states that if two matrices A and B are positive definite matrices, then

$$|A \circ B| \geq |B| \prod_i a_{ii},$$

where \circ denotes the Hadamard product of matrices. As a result

$$|A'_N \circ A'_N| = |A_N| \geq |A'_N| \prod_i a_{ii}.$$

Thus our matrix is invertible. We then have established formula (3.8), with an appropriate m .

Let k_1 be the weight of F . Since Δ does not vanish on H , we find from the valence formula applied to F that the sum of the orders of F at the different cusps of $\Gamma(N)$ is equal to

$$\frac{k_1}{12} [\Gamma(1) : \Gamma_1(N)].$$

On the other hand, the valence formula is also valid for the GMF f^{12} of weight $12k$ [8]. We then deduce from (3.8) that

$$k_1 = 12mk.$$

Letting $M = 12m$ we see that f^m/F is a GMF satisfying all the assumptions of Theorem 3.1. We conclude that $f^M = cF$, as required.

We now change one of the conditions in the above theorem from a condition on the order of the function at the cusps to a condition on the level N of the congruence subgroup.

Theorem 3.5 *Let f be a GMF of integral weight k on $\Gamma_1(N)$, and suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Suppose further that for the cusps a/c whose denominator does not divide N , the function will have the same order at a/c as any of those cusps whose denominators are (c, N) and that the rank of*

$$\left((\delta, c)^2 P_2(ag/(\delta, c)) \right)_{(\delta|N, 0 \leq g < \delta), (c|N, a)} \quad (3.9)$$

is equal to the number of cusps whose denominator divides N . Then f is a classical modular form.

Proof For given integers $r_{\delta, g}$ put

$$F(\tau) = \prod_{\delta|N} \prod_{0 \leq g < \delta} \eta_{\delta, g}(\tau)^{r_{\delta, g}},$$

We want to find $r_{\delta, g}$ such that $f^m = cF$ for some constant c . F is a modular form on $\Gamma_1(N)$ of weight $k_1 = \sum r_{\delta, 0}$ and by [20],

$$\text{ord}_{a/c} F = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0 \leq g < \delta} \frac{(\delta, c)^2}{\delta} P_2 \left(\frac{ag}{(\delta, c)} \right) r_{\delta, g}.$$

Using the condition (3.9), we have to solve now for $r_{\delta, g}$. Notice now that the product of two expressions whose Fourier coefficients are rational and p -integral for all but a finite number of primes has its Fourier coefficients to be rational and p -integral for all but a finite number of primes. $\eta_{\delta, g}(\tau)$ has rational Fourier coefficients that are p -integral for all but a finite number of primes. Thus the Fourier coefficients of F are rational and p -integral for all but a finite number of primes since F is a generalized eta product. We still want to show that $r_{\delta, g}$ can be chosen so that

$$\text{ord}_{a/c} F = mh_{a/c}, \quad (3.10)$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f . It is given that the rank of (3.9) is equal to the number of cusps whose denominator divides N . Thus we have a non trivial solution. Thus we have established formula (3.10), with an appropriate m . Since η does not vanish on H , we find from the valence formula applied to F that the sum of the orders of F at the different cusps of $\Gamma_1(N)$ is equal to

$$\frac{k_1}{12}[\Gamma(1) : \Gamma_1(N)].$$

On the other hand, the valence formula is also valid for the GMF f of weight k [8]. We then deduce from (3.10) that

$$k_1 = mk.$$

We see that f^m/F is a GMF satisfying all the assumptions of theorem 3.1. We conclude that $f^m = cF$, as required.

3.4 GMF's on $\Gamma(N)$ Representable as Eta-Products

A complete set of representatives of the cusps of $\Gamma(N)$ is given by:

$$\frac{a}{c} \tag{3.11}$$

where c is taken modulo N and a is taken modulo N and $(a, d = (N, c)) = 1$. In this set of representatives, the cusps pair up. The width of the every cusp $\frac{a}{c}$ in (3.11) is given by

$$w_{a/c} = N.$$

In the case of $\Gamma(N)$, we can also derive a theorem with strong restrictions at the order of the function at the cusps and then in a following theorem, we relax those conditions by imposing a condition on N as in the case of $\Gamma_1(N)$.

Theorem 3.6 *Let f be a GMF of integral weight k on $\Gamma(N)$. Suppose that the poles and zeroes of f are supported at the cusps and that the Fourier coefficients*

at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Suppose further that the order of the function f at each cusp a/c of $\Gamma(N)$ is independent of a and for the cusps a_1/c_1 whose denominator does not divide N , the function will have the same order as those cusps whose denominators are (c_1, N) . Then f is an eta quotient, i.e. there are integers $M \neq 0$ and m_t ($t \mid N$) such that

$$f^M(\tau) = c \prod_{t \mid N} \Delta(t\tau)^{m_t}.$$

Proof We have

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

For given integers m_t put

$$F(\tau) = \prod_{t \mid N} \Delta(t\tau)^{m_t}.$$

Then F is a modular form on $\Gamma(N)$ and

$$\text{ord}_{a/c} F = w_{a/c} \left(\sum_{t \mid N} \frac{(t, c)^2}{t} m_t \right).$$

Note that the order at every cusp a/c is independent of a . We want to show that m_t can be chosen so that

$$\text{ord}_{a/c} F = m h_{a/c} \tag{3.12}$$

for all cusps a/c of $\Gamma_1(N)$. Here $h_{a/c}$ is the order of f^{12} at a/c and m is an appropriate non-zero integer depending only on f . Note that by assumption $h_{a/c}$ is independent of a . It will also be sufficient to prove that the square matrix

$$A_N = ((t, c)^2)_{t \mid N, c \mid N}$$

of size $\sigma_0(N)$ is invertible. The above matrix is exactly the same matrix that appeared in the case of $\Gamma_1(N)$. Hence we have established formula (3.12), with an appropriate m .

Let k_1 be the weight of F . Since Δ does not vanish on H , we find from the valence formula applied to F that the sum of the orders of F at the different cusps of $\Gamma(N)$ is equal to

$$\frac{k_1}{12}[\Gamma(1) : \Gamma(N)].$$

On the other hand, the valence formula is also valid for the GMF f^{12} of weight $12k$ [8]. We then deduce from (3.12) that

$$k_1 = 12mk.$$

Letting $M = 12m$ we see that f^m/F is a GMF satisfying all the assumptions of Theorem 3.1. We conclude that $f^M = cF$, as required. We now change one of the conditions in the above theorem from a condition on the order of the function at the cusps to a condition on the level N of the congruence subgroup.

Theorem 3.7 *Let f be a GMF of integral weight k on $\Gamma(N)$, and suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p -integral for all but a finite number of primes p . Suppose further that for the cusps a/c whose denominator does not divide N , the function will have the same order at a/c as at any of those cusps whose denominators are (c, N) and that the rank of*

$$\left((\delta, c)^2 P_2(ag/(\delta, c)) \right)_{(\delta|N, 0 \leq g < \delta), (c|N, a)} \tag{3.13}$$

is equal to the number of cusps whose denominator divides N . Then f is a classical modular form.

Proof Follow exactly the proof of Theorem 3.5.

CHAPTER 4

Arithmetic Identities

4.1 Introduction

In this chapter, we determine arithmetic identities modulo 3, 7 and 4. The groups associated to the arithmetic identities are $\Gamma_0(3)$, $\Gamma_0(7)$ and $\Gamma_0(4)$. Notice that the three groups in question all have a fundamental region whose closure has genus 0, so there exists no nontrivial cusp forms of weight 2 and trivial multiplier system, i.e., $S_2(\Gamma_0(n)) = \{0\}$ for $n = 3, 4, 7$. Notice also that each of $\Gamma_0(3)$ and $\Gamma_0(7)$ has two cusps, so $M_2(\Gamma_0(3))$ and $M_2(\Gamma_0(7))$ have a basis consisting of one Eisenstein series while $M_2(\Gamma_0(4))$ has three cusps and hence $\dim M_2(\Gamma_0(4)) = 2$ (2 Eisenstein series). To determine the arithmetic identities, we need to define the following function

$$\delta_p(n) = \sum_{\substack{d|n \\ d=QR \pmod{p}}} 1 - \sum_{\substack{d|n \\ d \neq QR \pmod{p}}} 1 = \sum_{d|N} \chi(d)$$

where p is a prime and $\chi(d) = \left(\frac{d}{p}\right)$ is the quadratic character mod p , and QR stands for a quadratic residue mod p . $\delta_p(n)$ will appear naturally in the Fourier coefficients of Eisenstein series of weight 1. Then we square Eisenstein series of weight 1 to get an Eisenstein series of weight 2 which span $M_2(\Gamma_0(p))$ for $p = 3, 7$. For $n = 4$, Vestal [24] determined the basis of $M_2(\Gamma_0(4))$ explicitly. It turned out that the basis of this vector space is spanned by the logarithmic

derivative of the generalized Dedekind eta function for certain values of δ and g . By squaring the Eisenstein series of weight 1 on $\Gamma_0(4)$ and comparing its coefficients to the coefficients of the logarithmic derivative of the generalized eta function, we will be able to obtain some arithmetic identities modulo 4.

4.2 Arithmetic Identities Modulo 3

To determine the arithmetic identities modulo 3, notice that the space $M_2(\Gamma_0(3))$ has no cusp forms. As a result, the space $M_2(\Gamma_0(3))$ is generated by Eisenstein series.

Eisenstein series of weight 1 [11] is defined by

$$G_{1,\chi} = \sum_m \chi(m)G_{1,m}$$

where χ is a non trivial Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^*$ and $G_{1,m}$ is given by

$$G_{1,m} = \frac{m}{N} - \frac{1}{2} - \frac{q^m}{1-q^m} + \sum_{\nu=1}^{\infty} \left[\frac{q^{\nu N-m}}{1-q^{\nu N-m}} - \frac{q^{\nu N+m}}{1-q^{\nu N+m}} \right]$$

For our purposes, we need Eisenstein series of weight 1 in the following form and it is given by the following theorem [11].

Theorem 4.1 *Let χ be an odd character, i.e., $\chi(-1) = -1$. Then*

$$G_{1,\chi} = B_{1,\chi} - 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi(d)q^n$$

where $B_{m,\chi}$ ($m \in \mathbb{Z}, m \geq 0$) is defined by

$$\sum_{a=1}^c \chi(a) \frac{te^{at}}{e^{ct} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} \quad (4.1)$$

$q = e^{2\pi i\tau}, \tau \in H$.

Let us define our character here:

$$\chi_3(n) = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases},$$

From (4.1) and the Taylor expansion of e^x , it will be easy to see that $B_{1,\chi_3} = -\frac{1}{3}$. Thus G_{1,χ_3} can be written as:

$$G_{1,\chi_3} = -\frac{1}{3} - 2 \sum_{n=1}^{\infty} \delta_3(n) q^n$$

Now G_{1,χ_3} is a modular form of weight 1 also the multiplier system of G_{1,χ_3} is ± 1 [11]. When we square it, we get a modular form of weight 2 with a trivial multiplier system, since G_{1,χ_3} has a multiplier system of values ± 1 . As we mentioned before, $\dim M_2(\Gamma_0(3)) = 1$ so the basis consists of one Eisenstein series. It was shown by Hecke in 1927 that

$$E_2(z) = -\frac{\pi}{y} + \frac{\pi^2}{3} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}\right),$$

has the same transformation law as modular forms of weight 2 and trivial character under elements of $M_2(\Gamma(1))$ [7]. $E_2(z)$ is not holomorphic in H . However, we have the following theorem [14].

Theorem 4.2 $E_2(z) - pE_2(pz)$ is a holomorphic modular form of weight 2 on H with respect to $\Gamma_0(p)$ for any prime p .

Now applying Theorem 4.2, we see that

$$\begin{aligned} G_2(z) = E_2(z) - 3E_2(3z) &= \frac{\pi^2}{3} - \frac{3\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} + 24\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i 3n z} \\ &= \frac{-2\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} + 24\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i 3n z}. \end{aligned}$$

G_2 is a basis for modular forms of weight 2 since there are no cusp forms of weight 2 on $\Gamma_0(3)$. Thus $G_{1,\chi_3}^2 = cG_2$ where c is a complex number. Looking at the constant term of G_{1,χ_3} and G_2 , we find out that

$$G_{1,\chi_3}^2(z) = -\frac{1}{6\pi^2} G_2(z).$$

Thus

$$\left(-\frac{1}{3} - 2 \sum_{n=1}^{\infty} \delta_3(n) q^n\right)^2 = \frac{1}{9} + \frac{4}{3} \sum_{n=1}^{\infty} \sigma(n) q^n - 4 \sum_{n=1}^{\infty} \sigma(n) q^{3n}. \quad (4.2)$$

Lemma 4.1 For $n = 3^r m$, $r \neq 0$ and $(3, m) = 1$, we have

$$\frac{4}{3}\sigma(n) - 4\sigma(3^{r-1}m) = \frac{4}{3}\sigma(m). \quad (4.3)$$

Proof

$$\begin{aligned} \frac{4}{3}\sigma(3^r m) - 4\sigma(3^{r-1}m) &= \frac{4}{3}\sigma(3^r)\sigma(m) - 4\sigma(3^{r-1})\sigma(m) \\ &= \sigma(m)\left[\frac{4}{3}\left(\frac{3^{r+1}-1}{2}\right) - 4\left(\frac{3^r-1}{2}\right)\right] \\ &= \frac{4}{3}\sigma(m). \end{aligned}$$

Expanding the above series on the left, we get the following equations that Farkas obtained in [5].

Theorem 4.3 Let n be a positive integer. If $n \equiv 1$ or $n \equiv 2$ modulo 3, we have

$$\delta_3(n) + 3 \sum_{j=1}^{n-1} \delta_3(j)\delta_3(n-j) = \sigma(n).$$

If $n \equiv 0$ modulo 3, say $n = 3^r m$ with $(3, m)=1$, then

$$\delta_3(n) + 3 \sum_{j=1}^{n-1} \delta_3(j)\delta_3(n-j) = \sigma(m) = \sigma'(n),$$

where

$$\sigma'(n) = \sum_{d|n, 3 \nmid d} d.$$

Proof The first identity follows directly from (4.2) and the second from (4.2) and (4.3).

4.3 Arithmetic Identities modulo 7

To determine arithmetic identities modulo 7, we repeat the same process using the fact that the space $M_2(\Gamma_0(7))$ is also spanned by one Eisenstein series. We have to define a character modulo 7 to deduce identities similar to

the identities modulo 3. We now define our character to be Legendre symbol. Define

$$\chi_7(n) = \left(\frac{n}{7}\right) = \begin{cases} 1 & n = QR \pmod{7} \\ -1 & n \neq QR \pmod{7} \\ 0 & 7 \mid n \end{cases},$$

Now, G_{1,χ_7} is given by

$$G_{1,\chi_7} = -1 - 2 \sum_{n=1}^{\infty} \delta_7(n) q^n.$$

Similarly, we get

$$\begin{aligned} G_2(z) = E_2(z) - 7E_2(7z) &= \frac{\pi^2}{3} - \frac{7\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} + 56\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i 7n z} \\ &= -2\pi^2 - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} + 56\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i 7n z}. \end{aligned}$$

Now G_{1,χ_7} is a modular form of weight 1, so when we square it, we get a modular form of weight 2 with a trivial multiplier system. G_2 is a basis for modular forms of weight 2 since there are no cusp forms of weight 2 on $\Gamma_0(7)$. Thus $G_{1,\chi_7} = cG_2$ where c is a complex number. Looking at the constant term of G_{1,χ_7} and G_2 , we find out that

$$G_{1,\chi_7}^2(z) = -\frac{1}{2\pi^2} G_2(z).$$

Thus we get

$$\left(-1 - 2 \sum_{n=1}^{\infty} \delta_7(n) q^n\right)^2 = 1 + 4 \sum_{n=1}^{\infty} \sigma(n) q^n - 28 \sum_{n=1}^{\infty} \sigma(n) q^{7n}. \quad (4.4)$$

Lemma 4.2 For $n = 7^r m$, $r \neq 0$ where $(7, m) = 1$, we have

$$4\sigma(n) - 28\sigma(7^{r-1}m) = 4\sigma(m) \quad (4.5)$$

Proof

$$\begin{aligned}
4\sigma(7^r m) - 28\sigma(7^{r-1}m) &= 4\sigma(7^r)\sigma(m) - 28\sigma(7^{r-1})\sigma(m) \\
&= \sigma(m)\left[4\left(\frac{7^{r+1}-1}{6}\right) - 28\left(\frac{7^r-1}{6}\right)\right] \\
&= 4\sigma(m).
\end{aligned}$$

Theorem 4.4 *Let n be a positive integer, $n \not\equiv 0$ modulo 7, we have*

$$\delta_7(n) + \sum_{j=1}^{n-1} \delta_7(j)\delta_7(n-j) = \sigma(n).$$

If $n \equiv 0$ modulo 7, say $n = 7^r m$ with $(7, m) = 1$, then

$$\delta_7(n) + \sum_{j=1}^{n-1} \delta_7(j)\delta_7(n-j) = \sigma(m) = \sigma'(n)$$

where

$$\sigma'(n) = \sum_{d|n, 7 \nmid d} d.$$

This is another identity deduced by Farkas [5].

4.4 Arithmetic Identities Modulo 4

Let us define our character here.

$$\chi_4(n) = \left(\frac{-4}{n}\right) = \begin{cases} 1 & n \equiv 1 \pmod{4} \\ -1 & n \equiv -1 \pmod{4} \\ 0 & \text{otherwise} \end{cases},$$

Now we will consider the Eisenstein series of weight 1 on $\Gamma_0(4)$, when we square it we will get weight 2 Eisenstein series on $\Gamma_0(4)$. But $M_2(\Gamma_0(4))$ is two dimensional. In [24], Vestal determined the basis of $M_2(\Gamma_0(4))$ explicitly. We define now an arithmetic function which will appear in the Fourier expansion of the logarithmic derivative of the generalized Dedekind eta function. Define $\sigma^{\delta, g}$ by

$$\sigma^{(\delta, g)}(N) = \sum_{\substack{d|N \\ d \equiv g \pmod{\delta}}} d + \sum_{\substack{d|N \\ d \equiv -g \pmod{\delta}}} d$$

and define $\delta_{4,2}$ by

$$\delta_{4,2}(n) = \sum_{\substack{d|n \\ d \equiv 1 \pmod{4}}} 1 - \sum_{\substack{d|n \\ d \equiv -1 \pmod{4}}} 1.$$

To present the basis, we use generalized Dedekind η -function $\eta_{\delta,g}(\tau)$. Recall that

$$\eta_{\delta,g}(\tau) = e^{\pi i P_2(\frac{g}{\delta})\delta\tau} \prod_{\substack{m>0 \\ m \equiv g \pmod{\delta}}} (1 - x^m) \prod_{\substack{m>0 \\ m \equiv -g \pmod{\delta}}} (1 - x^m),$$

where $x = e^{2\pi i\tau}$, $\tau \in H$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} = t - [t]$ is the fractional part of t .

Vestal calculated the logarithmic derivative of $\eta_{\delta,g}(\tau)$:

$$\frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}(\tau)} = \pi i \delta P_2(g/\delta) - 2\pi i \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N) q^N.$$

For simplicity, let $H_2^{(\delta,g)}$ denote the normalization of the above series.

$$H_2^{(\delta,g)} = 1 - \frac{2}{\delta P_2(g/\delta)} \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N) q^N.$$

Then $H_2^{(\delta,g)(\tau)} \in M_2(\Gamma_0(\delta))$. Note that $M_2(\Gamma_0(4))$ is two dimensional, so the basis of $\Gamma_0(4)$ [24] consists of

$$H_2^{(4,1)} = 1 + 24 \sum_{N=1}^{\infty} \sigma^{(4,1)}(N) q^N$$

and

$$H_2^{(4,2)} = 1 + 6 \sum_{N=1}^{\infty} \sigma^{(4,2)}(N) q^N.$$

Vestal proceeds to find that

$$\theta^4(\tau) = \frac{1}{3} H_2^{(4,1)}(\tau) + \frac{2}{3} H_2^{(4,2)}(\tau),$$

where

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau}.$$

Notice that $\theta(\tau) \in M_{1/2}(\Gamma_0(4), \nu_\theta)$ and that $\nu_\theta^4 \equiv 1$. We relate now the above basis to our Eisenstein series of weight 1. With the character defined above, we have

$$G_{1,\chi} = -\frac{1}{2} - 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) q^n.$$

As a result, we get

$$G_{1,\chi} = -\frac{1}{2} - 2 \sum_{n=1}^{\infty} \delta_{4,2}(n) q^n,$$

where

$$\delta_{4,2} = \sum_{d|n} \chi(d).$$

Notice that $\delta_{4,2}$ is the difference between the number of divisors of n congruent to 1 mod 4 and the number of divisors of n congruent to -1 modulo 4. By a classical result that goes back to Jacobi

$$\theta^2(z) = 1 + 4 \sum_{n=1}^{\infty} \delta_{4,2} z^n.$$

Hence

$$\theta^2(\tau) = -2G_{1,\chi}(\tau).$$

Therefore

$$4G_{1,\chi}^2(\tau) = \frac{1}{3}H_2^{(4,1)}(\tau) + \frac{2}{3}H_2^{(4,2)}(\tau).$$

As a result, we get

$$4 \left(-\frac{1}{2} - 2 \sum_{n=1}^{\infty} \delta_{4,2}(n) q^n \right)^2 = \frac{1}{3}H_2^{(4,1)}(\tau) + \frac{2}{3}H_2^{(4,2)}(\tau).$$

This leads to the following identity

$$8\delta_{4,2}(n) + 16 \sum_{j=1}^{n-1} \delta_{4,2}(j)\delta_{4,2}(n-j) = 8\sigma^{(4,1)}(n) + 4\sigma^{(4,2)}(n).$$

Consequently,

$$2\delta_{4,2}(n) + 4 \sum_{j=1}^{n-1} \delta_{4,2}(j)\delta_{4,2}(n-j) = 2\sigma^{(4,1)}(n) + \sigma^{(4,2)}(n).$$

This is a new proof of another identity due to Farkas [6].

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