A Hecke Correspondence for Automorphic Integrals with Infinite Log-Polynomial Periods

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Austin Daughton
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Examining Committee Members:

Marvin Knopp, Advisor, Mathematics
Boris Datskovsky, Chair, Mathematics
Shiferaw Berhanu, Mathematics
Gerardo Mendoza, Mathematics
Wladimir Pribitkin, External Member, College of Staten Island, Mathematics

ABSTRACT

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Professor Boris A. Datskovsky, Chair

Since Hecke first proved his correspondence between Dirichlet series with functional equations and automorphic forms, there have been a great number of generalizations (see, for example, [3] [7] [9] [10]). Of particular interest is a generalization due to Bochner that gives a correspondence between Dirichlet series with any finite number of poles that satisfy the classical functional equation and automorphic integrals with (finite) log-polynomial sum period functions.

In this dissertation, we extend Bochner's result to Dirichlet series with finitely many essential singularities. With some restrictions on the underlying group and the weight, we also prove a correspondence for Dirichlet series with infinitely many poles. For this second correspondence, we provide a technique to approximate automorphic integrals with infinite log-polynomial sum period functions by automorphic integrals with finite log-polynomial period functions.

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Dedicated to the memory of Marvin Knopp and Leon Ehrenpreis

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CHAPTER 1

Introduction

1.1 Motivation

Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$ be an exponential series with $a_n = \mathcal{O}(n^\gamma)$ for some $\gamma > 0$. In 1936, Hecke proved in [4] that the transformation law $\overline{\upsilon}(T)z^{-k}f(-1/z) = f(z)$ under the inversion $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is equivalent to the following three conditions:

(1) The completed Dirichlet series $\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\sum_{n\geq 1} a_n n^{-s}$ has a meromorphic continuation to the entire s-plane with at most simple poles at 0 and k,

(2) $\Phi(s)$ satisfies the functional equation $\Phi(k-s) = i^k\upsilon(T)\,\Phi(s)$, and

(3) $\Phi(s) - a_0\left(\frac{i^k\upsilon(T)}{s-k} - \frac{1}{s}\right)$ is bounded in every vertical strip.

Thus we have a correspondence between automorphic forms and Dirichlet series with functional equations and certain growth conditions. Notice that in Hecke's correspondence, the Dirichlet series may have poles at 0 and k, and these poles are at worst simple. Using the same proof techniques, Bochner [2] generalized Hecke's result to allow for Dirichlet series with any finite number of poles. A slight reformulation of Bochner's result can be stated as follows.

Theorem 1.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$ be holomorphic in the upper half-*

plane with $a_n = \mathcal{O}(n^\gamma)$ *for some* $\gamma > 0$*. Let* $q(z) = \sum_{j=1}^{L} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^\ell$ *be a log-polynomial sum and define the functions*

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\Phi_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_f(s),$$

$$Q(s) = \sum_{j=1}^{L} \sum_{\ell=0}^{M(j)} \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} + a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s}\right).$$

Then the following are equivalent:

(A) $f(z)$ *satisfies the transformation law* $\overline{\upsilon}(T) z^{-k} f(-1/z) = f(z) + q(z)$*.*

(B) $\Phi_f(s) - Q(s)$ *has an analytic continuation to the entire s-plane that is bounded in vertical strips, and* $\Phi_f(k-s) = i^k \upsilon(T)\, \Phi_f(s)$*.*

This yields a correspondence between automorphic integrals with finite log-polynomial sum period functions and Dirichlet series with the classical functional equation and finitely many poles. The log-polynomial sum $q(z)$ is particularly important to this general correspondence, as the poles of the Dirichlet series are determined exactly by the form of the log-polynomial sum and vice-versa. More precisely, the poles of the completed Dirichlet series are restricted to the set $\{0, k, \alpha_j\}$, and the pole at α_j has order $M(j)+1$ with principal part given in terms of the coefficients $\delta(j,\ell)$.

Given this description of the singularities, by allowing either the sum on j or on ℓ to be an infinite sum, the corresponding Dirichlet series should have infinitely many poles or finitely many essential singularities respectively. We prove that these generalizations of Bochner's result hold for finitely many essential singularities in Theorem 3.8 and (with some restrictions on the group and the weight) for infinitely many poles in Theorems 4.3 and 4.4. To prove these results, we will provide good estimates on the 'infinite' log-polynomial sums as well as a new technique to estimate an automorphic integral with infinite log-polynomial periods by automorphic integrals with finite log-polynomial periods.

Specifically, in Chapter 2, we provide a rather explicit construction of generalized Poincaré series that will be needed to create automorphic integrals. Using the same techniques as Bochner and Hecke, we deal with the case of finitely many essential singularities in Chapter 3. Chapter 4 is dedicated to dealing with the case of infinitely many poles. We establish the desired correspondence for the theta group and some weights in this chapter.

1.2 Background and Definitions

Denote by $SL_2(\mathbb{R})$ the group of real 2×2 matrices with determinant 1. We let $SL_2(\mathbb{R})$ act on the upper half-plane $\mathcal{H}$ via linear fractional transformations: $Mz = \frac{az+b}{cz+d}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We call a transformation $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ parabolic, elliptic, or hyperbolic if $|\operatorname{tr}(M)| = |a+d|$ is equal to, smaller than, or larger than 2, respectively.

Definition 1.2. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. A **fundamental region** $\mathcal{R}$ for Γ is a domain in $\mathcal{H}$ such that*

1. *No two points of $\mathcal{R}$ are equivalent under the action of Γ, and*

2. *Every point in $\mathcal{H}$ is equivalent with respect to Γ to some point in $\overline{\mathcal{R}}$.*

*A **parabolic point** (or parabolic cusp) for Γ in $\mathcal{R}$ is any point $q \in \mathbb{R} \cup \{i\infty\}$ such that q is in the closure of $\mathcal{R}$ with respect to the topology of the Riemann sphere and q is fixed by a non-identity parabolic transformation in Γ.*

Definition 1.3. *An **automorphic form** on the group Γ of weight k is a function $f(z)$, meromorphic in $\mathcal{H}$, which satisfies certain growth conditions and the transformation law*

$$f(Mz) = \upsilon(M)(cz+d)^k f(z) \tag{1.1}$$

for every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. (Here $\upsilon(M)$ is a complex number of modulus 1 that is independent of z.)

*We say that a meromorphic function f is an **automorphic function** on the group Γ if*

$$f(Mz) = f(z)$$

for all $M \in \Gamma$.

We fix the branch of $(cz+d)^k$ by the convention that

$$-\pi \leq \arg(z) < \pi$$

for $z \neq 0$. It is easy to see that if there exists a function $f(z) \not\equiv 0$ satisfying (1.1), then the function υ must satisfy

$$\upsilon(M_3)(c_3z+d_3)^k = \upsilon(M_1)\upsilon(M_2)(c_1M_2z+d_1)^k(c_2z+d_2)^k \tag{1.2}$$

for every $z \in \mathcal{H}$ and $M_1 = \left(\begin{smallmatrix} * & * \\ c_1 & d_1 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} * & * \\ c_2 & d_2 \end{smallmatrix}\right)$, and $M_3 = M_1M_2 = \left(\begin{smallmatrix} * & * \\ c_3 & d_3 \end{smallmatrix}\right)$ in Γ. Condition (1.2) is called the *consistency condition.*

Definition 1.4. *We say that a function $\upsilon\colon \Gamma \to \mathbb{C}$ is a **multiplier system** for Γ in weight k provided that υ is of absolute value* 1 *and satisfies the consistency condition* (1.2)*.*

The consistency condition implies that $\upsilon(I) = 1$ and $\upsilon(-I) = \pm e^{\pi ik}$ whenever $-I \in \Gamma$. We shall assume that $\upsilon(-I) = e^{\pi ik}$. This condition is called the "nontriviality condition" because there are no nontrivial forms for which $\upsilon(-I) = -e^{\pi ik}$.

Set $S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and denote by $G(\lambda)$ the group generated by these two matrices. These are called the *Hecke groups.* Two important examples are the full modular group $SL_2(\mathbb{Z}) = G(1)$ and the theta group $\Gamma_\vartheta = G(2)$, a subgroup of index 3 in $SL_2(\mathbb{Z})$.

Because S_λ and T generate the Hecke group $G(\lambda)$, a multiplier system for $G(\lambda)$ can be completely determined from its values on S_λ and T and the consistency condition. From the consistency and nontriviality conditions, we can easily deduce that $\upsilon(T) = \pm e^{-\pi ik/2} = \pm i^{-k}$.

We define the usual "slash" operator $|_{\upsilon}^{k}$ by

$$(f|_{\upsilon}^{k}M)(z) = \overline{\upsilon}(M)(cz+d)^{-k}f(Mz) \tag{1.3}$$

for $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$. When k and υ are clear from context, we shall simply write $f|M$.

Definition 1.5. *An* ***automorphic integral*** *on the group* Γ *of weight* k *and multiplier* υ *is a holomorphic function* f *satisfying certain growth conditions and such that*

$$(f|_{\upsilon}^{k}M)(z) = f(z) + q_M(z) \tag{1.4}$$

for every $M \in \Gamma$ *and* $z \in \mathcal{H}$. *The functions* $q_M(z)$ *are called the period functions (or cocycles) associated to* f.

We call a collection of functions $\{q_M(z) \colon M \in \Gamma\}$ ***parabolic cocycles*** *if*

$$q_{M_1M_2}(z) = (q_{M_1}|_{\upsilon}^{k}M_2)(z) + q_{M_2}(z) \tag{1.5}$$

for all $M \in \Gamma$. *This condition is called the cocycle condition.*

Note that if f is a non-zero function satisfying (1.4), then (1.5) follows from the consistency condition (1.2). Also, given functions $q_{S_\lambda}(z)$ and $q_T(z)$, we can generate a collection of parabolic cocycles $\{q_M(z)\}$ on $G(\lambda)$ by repeatedly applying the cocycle condition to a word in the generators S_λ and T. (Note that the functions q_{S_λ} and q_T cannot be completely arbitrary. There are restrictions depending on the relations in the group.) Often, we will be concerned only with automorphic integrals $f(z)$ where $q_{S_\lambda} \equiv 0$; in these cases, the collection of period functions depends only upon q_T, and so we shall call q_T the period function for $f(z)$.

CHAPTER 2

Generalized Poincaré Series

2.1 The Space $\mathcal{P}$

Throughout this chapter, let $z = x + iy$. We denote by $\mathcal{P}$ the space of functions f holomorphic in $\mathcal{H}$ such that

$$|f(z)| \leq K(|z|^A + y^{-B}) \tag{2.1}$$

for some $K, A, B > 0$. While this space is defined by a growth restriction at the boundary of $\mathcal{H}$, it is quite large and well suited for our purposes. As we shall see in the following section, the generalized Poincaré series will converge when the period function is in $\mathcal{P}$. We are therefore interested in being able to show that certain functions are in $\mathcal{P}$. Furthermore, we shall need to have an exact formula for the constants K, A, and B. For this, we shall often use the following lemma.

Lemma 2.1. *Let $K_j, M_j, A_j, B_j \geq 0$ for $j = 1, 2$. Then there exist $K, A, B > 0$ such that*

$$\sum_{j=1}^{2} (K_j|z|^{A_j} + M_j y^{-B_j}) \leq K(|z|^A + y^{-B})$$

for all $z \in \mathcal{H}$.

Proof. Let $K^* = \max\limits_{1\leq j\leq 2}(K_j, M_j, 1)$, $A = \max(A_1, A_2, 1), B = \max(B_1, B_2, 1)$.

Then we have $\sum_{j=1}^{2}(K_j|z|^{A_j} + M_j y^{-B_j}) \leq K^*(|z|^{A_1} + |z|^{A_2} + y^{-B_1} + y^{-B_2})$. Let $h(z) = \frac{|z|^{A_1}+|z|^{A_2}+y^{-B_1}+y^{-B_2}}{|z|^A+y^{-B}}$. We will show that h is bounded in $\mathcal{H}$. To that end, write $\mathcal{H} = S_1 \cup S_2 \cup S_3$ where $S_1 = \{\mathrm{Im}(z) > 1\}$, $S_2 = \{|z| \leq 1\}$, $S_3 = \{|z| > 1, 0 < \mathrm{Im}(z) \leq 1\}$.

For $z \in S_1$, $|z| > 1$ and $0 < y^{-B_j}, y^{-B} < 1$. Then

$$h(z) \leq \frac{|z|^{A_1} + |z|^{A_2} + 2}{|z|^A} = |z|^{A_1 - A} + |z|^{A_2 - A} + 2|z|^{-A} \leq 4.$$

For $z \in S_2$, $0 < |z| \leq 1$ and $0 < y \leq 1$. Thus

$$h(z) \leq \frac{2 + y^{-B_1} + y^{-B_2}}{y^{-B}} = 2y^B + y^{B-B_1} + y^{B-B_2} \leq 4.$$

For $z \in S_3$, $|z| > 1$, $0 < y \leq 1$, $A \geq A_1, A_2$, and $B \geq B_1, B_2$. So

$$\begin{aligned} h(z) &= \frac{|z|^{A_1} + |z|^{A_2}}{|z|^A + y^{-B}} + \frac{y^{-B_1} + y^{-B_2}}{|z|^A + y^{-B}} \\ &\leq \frac{|z|^{A_1} + |z|^{A_2}}{|z|^A} + \frac{y^{-B_1} + y^{-B_2}}{y^{-B}} \\ &= |z|^{A_1 - A} + |z|^{A_2 - A} + y^{B - B_1} + y^{B - B_2} \\ &\leq 4. \end{aligned}$$

Hence $h(z) \leq 4$ in $\mathcal{H}$. □

Note that if $A_1 > 0$ or $A_2 > 0$, then $A = \max(A_1, A_2)$ will work in the lemma. The same is true for B; for K, however, we need to use 4 times the max. Using the same proof technique, we can actually prove the following general result.

Corollary 2.2. *Let $K_j, M_j, A_j, B_j \geq 0$ for $j = 1, \ldots, n$. Then there exist $K, A, B > 0$ such that*

$$\sum_{j=1}^{n}(K_j|z|^{A_j} + M_j y^{-B_j}) \leq K(|z|^A + y^{-B})$$

for all $z \in \mathcal{H}$. Furthermore, if $A_j > 0$ for any j, then $A = \max(A_1, \ldots, A_n)$ works, and similarly, if $B_j > 0$ for any j, then $B = \max(B_1, \ldots, B_n)$ works. If $K_j > 0$ or $M_j > 0$ for any j, then $K = 2n \max_{1 \leq j \leq n}(M_j, K_j)$ works.

Another easy but rather useful consequence of the lemma is that $\mathcal{P}$ has a number of nice closure properties.

Corollary 2.3. *The space $\mathcal{P}$ is closed under the slash operator, addition, and multiplication.*

Proof. Suppose that $f, g \in \mathcal{P}$ with $|f(z)| \leq K(|z|^A + y^{-B})$ and $|g(z)| \leq M(|z|^C + y^{-D})$. Lemma 2.1 immediately tells us that $f + g \in \mathcal{P}$. For multiplication, we combine Corollary 2.2 with the fact that $ab \leq \frac{1}{2}(a^2 + b^2)$ for any $a, b \in \mathbb{R}$:

$$\begin{aligned}
|f(z)g(z)| &\leq KM(|z|^{A+C} + y^{-(B+D)} + |z|^A y^{-D} + |z|^C y^{-B}) \\
&\leq KM\left(|z|^{A+C} + y^{-(B+D)} + \frac{1}{2}(|z|^{2A} + y^{-2D} + |z|^{2C} + y^{-2B})\right) \\
&\leq 6KM\left(|z|^{\max(A+C,2A,2C)} + y^{-\max(B+D,2B,2D)}\right) \\
&= K^*\left(|z|^{2\max(A,C)} + y^{-2\max(B,D)}\right).
\end{aligned}$$

Thus $fg \in \mathcal{P}$.

For the slash operator, note that $\mathrm{Im}(Mz) = \frac{y}{|cz+d|^2}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. Then we have that

$$\begin{aligned}
|f|_v^k M(z)| &= |v(M)(cz+d)^{-k} f(Mz)| \\
&\leq K|cz+d|^{-k}\left(\tfrac{|az+b|^A}{|cz+d|^A} + y^{-B}|cz+d|^{2B}\right).
\end{aligned}$$

Because $\mathcal{P}$ is closed under multiplication and addition, it is enough to show that $(\alpha z + \beta)^C \in \mathcal{P}$ for any real numbers α, β, C. If $\alpha = 0$, then this is clear. So assume $\alpha \neq 0$. For $C \geq 0$, we have

$$\begin{aligned}
|\alpha z + \beta|^C &\leq (|\alpha z| + |\beta|)^C \\
&\leq K'(|\alpha z|^C + |\beta|^C) \\
&\leq K''(|z|^C + y^{-1})
\end{aligned}$$

for some positive constants K', K''. For $C < 0$, $|\alpha z + \beta| \geq |\alpha y|$, and thus $|\alpha z + \beta|^C \leq |\alpha y|^C = |\alpha|^C y^C$. $\square$

2.2 Construction and Convergence

The construction of generalized Poincaré series for general H-groups (that is, finitely generated discrete subgroups of $SL_2(\mathbb{R})$ that contain translations and have the entire real line as their limit set) has been well studied in [6]. In particular, this construction works for all discrete Hecke groups with $\lambda \leq 2$. When $\lambda > 2$, the group $G(\lambda)$ is no longer an H-group and the construction no longer works. However, Knopp and Sheingorn have provided a slightly different construction in [8] for these groups.

We shall present here a more explicit construction in the special case of the theta group $\Gamma_\vartheta = G(2)$. Recall that the theta group is generated by $S_2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This group has just one relation: $T^2 = -I$. We can generate a collection of parabolic cocycles $\{q_M(z) \colon M \in \Gamma_\vartheta\}$ via the cocycle condition (1.5) given any functions q_{S_2} and q_T, subject to the restriction that $q_T|_\upsilon^k T + q_T = 0$.

In the following, we shall assume that $q_{S_2} \equiv 0$ and that $q_T \in \mathcal{P}$ with

$$|q_T(z)| \leq K(|z|^A + y^{-B}). \tag{2.2}$$

Fix $k \in \mathbb{R}$ and υ a multiplier system in weight k for Γ_ϑ. Assume that $q_T|_\upsilon^k T + q_T = 0$ and construct parabolic cocycles $\{q_M(z)\}$. Now suppose that w is a multiplier system for Γ_ϑ of weight m. Let $(\Gamma_\vartheta)_\infty = \langle S_2 \rangle$ be the 'stabilizer' of ∞ in Γ_ϑ. (The stabilizer of infinity in Γ_ϑ is more properly $\langle S_2, -I \rangle$. However, the 'stabilizer' given is the one required to make the sum well-defined and have the correct transformation laws.) Then we define the *generalized Poincaré series* $\Psi(\{q_M\}, m, w; z) = \Psi(z)$ by

$$\Psi(z) = \sum_{M \in (\Gamma_\vartheta)_\infty \backslash \Gamma_\vartheta} \frac{q_M(z)}{w(M)(cz+d)^m}. \tag{2.3}$$

First note that this sum is really over all distinct lower rows c, d of the elements in Γ_ϑ, and we can rewrite the equation as

$$\Psi(z) = \sum_{\substack{c,d \\ (c,d)=1 \\ c+d \text{ odd}}} \frac{q_M(z)}{w(M)(cz+d)^m}$$

where $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ is any element of Γ_ϑ with lower row c, d. The assumption that $q_{S_2} \equiv 0$ insures that the individual terms in the sum are independent of the choice of M.

This series converges absolutely and uniformly on compact subsets of $\mathcal{H}$ for m large. Specifically, we have the following result.

Proposition 2.4. *Let $\beta = \max(A/2, B+k/2)$. Then for $m > 2\beta+4$, the generalized Poincaré series* (2.3) *converges absolutely and uniformly on compact subsets of $\mathcal{H}$. Furthermore, $\Psi \in \mathcal{P}$ with*

$$|\Psi(z)| \leq KK^* \left(\frac{1+4|z|^2}{y^2}\right)^{m/2} (|z|^{6\beta+2k} + y^{-6\beta-2k}),$$

where K^* is a positive constant depending on m.

This is essentially Proposition 7 in [6]. Assuming then that m is large, it follows from absolute convergence that for any $V = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\vartheta$,

$$(\Psi|_v^k V)(z) = w(V)(cz+d)^m \Psi(z) - w(V)(cz+d)^m E_{m,w}(z) q_V(z) \tag{2.4}$$

where $E_{m,w}(z)$ is the Eisenstein series

$$E_{m,w}(z) = \sum_{\substack{c,d \\ (c,d)=1 \\ c+d \text{ odd}}} \overline{w}(M)(cz+d)^{-m}. \tag{2.5}$$

At this point we shall assume that $w \equiv 1$ and m is a large even integer. Then $E_{m,w}(z)$ is, in fact, an entire form on Γ_ϑ with $E_{m,w}(i\infty) = 2$ (and therefore not identically zero). Now the function

$$H(z) = -\frac{\Psi(z)}{E_{m,w}(z)}$$

satisfies the transformation law

$$(H|_v^k V)(z) = H(z) + q_V(z). \tag{2.6}$$

However, this is not quite enough to call $H(z)$ an automorphic integral since H may have poles in the upper half-plane. These poles are restricted to the zeros of the Eisenstein series $E_{m,w}$, of which there are finitely many in a fundamental region. We shall eliminate these poles while retaining the correct transformation law by subtracting an automorphic form of weight k with multiplier system υ whose poles have principal parts matching those of $H(z)$ exactly. We can do this because there is a Mittag-Leffler principle for automorphic forms on Γ_ϑ.

Theorem 2.5. *Let $\mathcal{R}_\vartheta = \{z \in \mathcal{H}\colon |z| > 1, |\operatorname{Re}(z)| < 1\}$ be a fundamental region for Γ_ϑ. Set*

$$\widetilde{\mathcal{R}_\vartheta} = (\overline{\mathcal{R}_\vartheta} \cap \mathcal{H}) \setminus (\{z\colon \operatorname{Re}(z) = 1\} \cup \{z\colon |z| = 1, 0 \le \operatorname{Re}(z) \le 1\}).$$

Assume that $z_1, \ldots, z_N$, $N \ge 0$ are distinct points in $\widetilde{\mathcal{R}_\vartheta}$, and let $\alpha_{j,t} \in \mathbb{C}$ $(1 \le t \le N,\ 1 \le j \le \ell(t))$ with $\alpha_{\ell(t),t} \ne 0$ for all t. Suppose further that $\beta_1, \ldots, \beta_\rho, \gamma_1, \ldots, \gamma_\mu, \delta_1, \ldots, \delta_\nu \in \mathbb{C}$ for some $\rho, \mu, \nu \ge 0$ such that $\beta_\rho, \gamma_\mu, \delta_\nu \ne 0$ (provided that $\rho > 0$, $\mu > 0$, $\nu > 0$ respectively). Then the following hold.

1. *(MS υ_k^+) If $k \ge 0$, there exists an automorphic form $F(z)$ on Γ_ϑ of weight k and multiplier system υ_k^+ such that F is holomorphic in $\widetilde{\mathcal{R}_\vartheta} \setminus \{z_1, \ldots, z_N\}$ with principal part*

$$\frac{\alpha_{\ell(t),t}}{(z - z_t)^{\ell(t)}} + \ldots + \frac{\alpha_{1,t}}{(z - z_t)} \tag{2.7}$$

at z_t for $1 \le t \le N$. Furthermore, F has expansions

$$(z+i)^{-k}\left(\beta_\rho \tau^{-2\rho} + \beta_{\rho-1}\tau^{-2(\rho-1)} + \ldots + \beta_1 \tau^{-2} + \sum_{n=0}^{\infty} b_n \tau^{2n}\right) \tag{2.8}$$

at $z = i$,

$$\gamma_\mu e^{-\pi i \mu z} + \ldots + \gamma_1 e^{-\pi i z} + \sum_{n=0}^{\infty} c_n e^{\pi i n z} \tag{2.9}$$

at $i\infty$, and

$$(z+1)^{-k}\left(\delta_\nu \omega^{-\nu+\kappa_1} + \ldots + \delta_1 \omega^{-1+\kappa_1} + \sum_{n=0}^{\infty} d_n \omega^{n+\kappa_1}\right) \tag{2.10}$$

at -1. (Here $\tau = \frac{z-i}{z+i}$, $\omega = e^{-2\pi i/(z+1)}$, and $\kappa_1 = \frac{k}{4} - [\frac{k}{4}]$.)

2. (MS υ_k^-) If $k > 2$, then there exists an automorphic form $G(z)$ on Γ_ϑ of weight k and multiplier system υ_k^- such that G is holomorphic in $\widetilde{\mathcal{R}_\vartheta} \setminus \{z_1, \ldots, z_N\}$ with principal part (2.7) at z_t. Moreover, $G(z)$ has the expansion (2.9) at $i\infty$ and expansions

$$(z+i)^{-k}\left(\beta_\rho \tau^{-2\rho+1} + \ldots \beta_1 \tau^{-1} + \sum_{n=0}^{\infty} b_n \tau^{2n+1}\right) \tag{2.11}$$

at $z = i$ and

$$(z+1)^{-k}\left(\delta_\nu \omega^{-\nu+\kappa_2} + \ldots + \delta_1 \omega^{-1+\kappa_2} + \sum_{n=0}^{\infty} d_n \omega^{n+\kappa_2}\right) \tag{2.12}$$

at $z = -1$. (Here $\kappa_2 = \frac{k+2}{4} - [\frac{k+2}{4}]$.)

Remarks. 1. The multiplier systems $\upsilon_k^\pm$ are defined by $\upsilon_k^\pm(S_2) = 1$ and $\upsilon_k^\pm(T) = \pm i^{-k}$. These are the only multiplier systems on Γ_ϑ with $\upsilon(S_2) = 1$. The value of κ_1 is defined by $\upsilon_k^+(S_2^{-1}T^{-1}) = e^{2\pi i \kappa_1}$ with $0 \le \kappa_1 < 1$ and can be determined using the consistency condition. Similarly, κ_2 is defined by $\upsilon_k^-(S_2^{-1}T^{-1}) = e^{2\pi i \kappa_2}$ with $0 \le \kappa_2 < 1$.

2. Because i is a fixed point of order 2 for Γ_ϑ, the expansion at $z = i$ is actually an expansion in the 'local uniformizing variable' $\tau^2 = (\frac{z-i}{z+i})^2$. Similarly, $e^{\pi i z}$ and ω are the local uniformizing variables at the cusps $i\infty$ and -1 respectively.

Before tackling the proof of Theorem 2.5, we need to show that there is a Mittag-Leffler theorem for modular functions on Γ_ϑ. (Recall that a meromorphic function $f(z)$ is called a modular function on Γ if $f(Mz) = f(z)$ for all $M \in \Gamma$.)

Proposition 2.6. *Let $\widetilde{\mathcal{R}_\vartheta}$, τ, and ω be as in Theorem 2.5. Assume that $z_1, \ldots, z_N$, $N \ge 0$ are distinct points in $\widetilde{\mathcal{R}_\vartheta}$, and let $\alpha_{j,t} \in \mathbb{C}$ ($1 \le t \le N$, $1 \le j \le \ell(t)$) with $\alpha_{\ell(t),t} \ne 0$ for all t. Suppose further that $\beta_1, \ldots, \beta_\rho$, $\gamma_1, \ldots, \gamma_\mu$, $\delta_1, \ldots, \delta_\nu \in \mathbb{C}$ for some $\rho, \mu, \nu \ge 0$ such that $\beta_\rho, \gamma_\mu, \delta_\nu \ne 0$ (provided that $\rho > 0$, $\mu > 0$, $\nu > 0$ respectively). Then there exists a modular function $f(z)$*

on Γ_ϑ *with principal part* (2.7) *at* z_t*, and* $f(z)$ *has expansions with principal part*

$$\beta_\rho \tau^{-2\rho} + \beta_{\rho-1}\tau^{-2(\rho-1)} + \ldots \beta_1 \tau^{-2} \tag{2.13}$$

at $z = i$*,*

$$\gamma_\mu e^{-\pi i \mu z} + \ldots + \gamma_1 e^{-\pi i z} \tag{2.14}$$

at $i\infty$*, and*

$$\delta_\nu \omega^{-\nu} + \ldots + \delta_1 \omega^{-1} \tag{2.15}$$

at -1*.*

Proof. We shall construct $f(z)$ using the so called 'Hauptmodul' for Γ_ϑ:

$$\varphi_0(z) = \left(\frac{\vartheta(z)}{\eta(z)}\right)^{12}.$$

Here $\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$ is the Jacobi theta function and $\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$ is the Dedekind eta function. Using the well known transformation laws for ϑ and η (see e.g. [5, pp. 39-48]), it is easily seen that φ_0 is a modular function for Γ_ϑ. That is, $\varphi_0(Mz) = \varphi_0(z)$ for any $M \in \Gamma_\vartheta$. Furthermore, $\varphi_0(z)$ is holomorphic and nonzero in the entire upper half-plane, but it has a zero of order 1 at the cusp -1 and a simple pole at the cusp $i\infty$.

To deal with $i\infty$, because $\varphi_0(z)$ has a simple pole, we take an appropriate linear combination $f(z; i\infty)$ of $\varphi_0, \ldots, \varphi_0^\mu$ to yield a modular function with principal part (2.14). Similarly, for the cusp -1, $\varphi_0^{-1}(z)$ is a modular function with a simple pole at -1 that is holomorphic in $\mathcal{H}$. So we can take an appropriate linear combination $f(z; -1)$ of $\varphi_0^{-1}, \ldots, \varphi_0^{-\nu}$ to get a modular function with principal part (2.15).

For a point z_t, consider the function $\varphi_0(z) - \varphi_0(z_t)$. This function has a simple pole at $i\infty$, a zero of order 1 at $z = z_t$, and is nonzero in $\widetilde{\mathcal{R}_\vartheta}$ as well as at i, -1. Then $(\varphi_0(z) - \varphi_0(z_t))^{-1}$ has a simple pole at z_t but is otherwise holomorphic in $\widetilde{\mathcal{R}_\vartheta} \cup \{i, i\infty, -1\}$. Again, an appropriate linear combination $f(z; z_t)$ of $(\varphi_0(z) - \varphi_0(z_t))^{-1}, \ldots, (\varphi_0(z) - \varphi_0(z_t))^{-\ell(t)}$ is a modular function with principal part (2.7).

The argument at $z = i$ is similar. An appropriate linear combination $f(z; i)$ of the functions $(\varphi_0(z) - \varphi_0(i))^{-1}, \ldots, (\varphi_0(z) - \varphi_0(i))^{-\rho}$ is a modular function with principal part (2.13).

Therefore $f(z) = f(z; i\infty) + f(z; -1) + f(z; i) + \sum_{t=1}^{N} f(z; z_t)$ is the desired modular function. $\square$

Proof of Theorem 2.5.

1. Note first that $\vartheta(z)$ is a modular form of weight $\frac{1}{2}$ on Γ_ϑ that is nonzero in $\mathcal{H}$. Thus $\vartheta^{2k}(z)$ is holomorphic and nonzero for any real number $k \geq 0$. Furthermore, $\vartheta^{2k}(z)$ is a modular form of weight k on Γ_ϑ. If we let υ_ϑ denote the multiplier system for ϑ, then $\upsilon_\vartheta^{2k}(S_2) = 1$ and $\upsilon_\vartheta^{2k}(T) = i^{-k}$. Hence the multipliers υ_ϑ^{2k} and υ_k^+ are exactly the same.

   We can thus construct the desired modular function $F(z)$ by using Proposition 2.6 to choose an appropriate modular function $f(z)$ and setting $F(z) = \vartheta^{2k}(z)f(z)$. Because ϑ^{2k} is holomorphic and nonzero in $\mathcal{H}$ and also at $i\infty$, we let $f(z)$ have principal part

$$\frac{\alpha_{\ell(t),t}}{\vartheta^{2k}(z_t)} \frac{1}{(z - z_t)^{\ell(t)}} + \ldots + \frac{\alpha_{1,t}}{\vartheta^{2k}(z_t)} \frac{1}{(z - z_t)}$$

   at z_t. The expansions at i and $i\infty$ are handled similarly. At the cusp -1, $\vartheta(z)$ is zero. We must therefore give $f(z)$ a pole of higher order to cancel the zero of $\vartheta^{2k}(z)$. Specifically, ϑ^{2k} has an expansion

$$\vartheta^{2k}(z) = (z+1)^{-k} \sum_{n=M}^{\infty} d_n \omega^{n+\kappa_1}$$

   where $M + \kappa_1 > 0$ and $d_M \neq 0$. We then give $f(z)$ the principal part

$$(z+1)^{-k} \left(\frac{\delta_\nu}{d_M} \omega^{\nu+M} + \ldots + \frac{\delta_1}{d_M} \omega^{1+M} \right)$$

   at -1.

2. For the multiplier system υ_k^-, we use the Eisenstein series $E_{k,\upsilon_k^-}(z)$ defined in (2.5) instead of ϑ^{2k}. The Eisenstein series E_{k,υ_k^-} converges exactly when $k > 2$, and in that case E_{k,υ_k^-} is a modular form of weight k

on Γ_ϑ with multiplier system υ_k^-. Then as above, we shall construct $G(z)$ by choosing an appropriate modular function $g(z)$ and setting $G(z) = E_{k,\upsilon_k^-}(z)g(z)$. However, the Eisenstein series will have zeros in $\mathcal{H}$ (and possibly at -1) for k large. If these zeros coincide with any of the points z_t or with i, then we give $g(z)$ a higher order pole (just as we did with ϑ^{2k} at -1) so that $G(z)$ has the correct principal part. $\square$

With this, we finally return to $H(z) = -\frac{\Psi(z)}{E_{m,w}(z)}$. Assume now either that $k \geq 0$ and $\upsilon = \upsilon_k^+$ or that $k > 2$ and $\upsilon = \upsilon_k^-$. Let $g(z)$ be the modular form of weight k whose principal parts match exactly the principal parts of $H(z)$ given by Theorem 2.5. Then the function $F(z) = H(z) - g(z)$ is holomorphic in $\mathcal{H}$. Because $g(z)$ is a modular form (and therefore $g|_\upsilon^k M = g$), from (2.6), we see that F satisfies the transformation law

$$(F|_\upsilon^k M)(z) = F(z) + q_M(z).$$

Together with the fact that $F(z)$ is holomorphic at $i\infty$ and -1, this transformation law implies that we have expansions

$$F(z) = \sum_{n=0}^{\infty} a_n e^{\pi i n z}$$

at $i\infty$ and

$$F(z) = \rho(z) + (z+1)^{-k} \sum_{n=0}^{\infty} b_n e^{2\pi i (n+\kappa)\frac{-1}{z+1}}$$

at -1, where $\rho(z) \in \mathcal{P}$ satisfies $q_{S_2^{-1}T^{-1}} = \rho|(S_2^{-1}T^{-1}) - \rho$ and κ is defined by $\upsilon_k^{\pm}(S_2^{-1}T^{-1}) = e^{2\pi i \kappa}$, $0 \leq \kappa < 1$. These expansions together actually imply that $F(z) \in \mathcal{P}$ (the details of this argument can be found in [6, pp. 622-3]). Furthermore, the coefficients a_n satisfy the growth condition $a_n = \mathcal{O}(n^\gamma)$ for some $\gamma > 0$. Both γ and the implied constant can be chosen to depend upon k, m, and the constants K, A, and B from (2.2).

CHAPTER 3

Essential Singularities

3.1 Convergence

Consider the following "infinite" version of a log-polynomial sum:

$$q(z) = \sum_{j=1}^{M} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{\infty} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell} \tag{3.1}$$

where the $\{\alpha_j\}$ are all distinct complex numbers, $\delta(j,\ell) \in \mathbb{C}$, and $z \in \mathcal{H}$. In this section, we would like to establish assumptions which will guarantee that such series will converge absolutely and uniformly on compact subsets of the upper half-plane. Furthermore, we want an infinite log-polynomial sum to be in the space $\mathcal{P}$.

For this, we need to estimate $|\log(z/i)|$. Let $\varepsilon > 0$. Then it is easy to establish that for real x

$$|\log x| \leq \begin{cases} \frac{1}{e\varepsilon} x^{\varepsilon} & \text{if } x \geq 1 \\ \frac{1}{e\varepsilon} x^{-\varepsilon} & \text{if } 0 < x < 1 \end{cases}$$

Using these inequalities, we can then show that

$$\frac{|\log x| x^{\varepsilon}}{(1+x)^{2\varepsilon}} \leq \begin{cases} \dfrac{1}{e\varepsilon} \left(\dfrac{x^2}{(1+x)^2}\right)^{\varepsilon} & \text{if } x \geq 1 \\ \dfrac{1}{e\varepsilon} \left(\dfrac{1}{(1+x)^2}\right)^{\varepsilon} & \text{if } 0 < x < 1 \end{cases}$$

Because $\frac{x^2}{(1+x)^2} < 1$ for $x \geq 1$ and $\frac{1}{(1+x)^2} < 1$ for $0 < x < 1$, we have

$$\frac{|\log x| x^{\varepsilon}}{(1+x)^{2\varepsilon}} \leq \frac{1}{e\varepsilon}$$

for any $x > 0$. That is,

$$|\log x| \leq \frac{1}{e\varepsilon}(1+x)^{2\varepsilon} x^{-\varepsilon}. \tag{3.2}$$

Since $(1+x)^2 x^{-1} \geq 1$, we have that

$$\begin{aligned} \left|\log \frac{z}{i}\right| &\leq |\log |z|| + |\arg(z/i)| \\ &\leq \frac{1}{e\varepsilon}(1+|z|)^{2\varepsilon}|z|^{-\varepsilon} + \frac{\pi}{2} \\ &\leq \left(\frac{1}{e\varepsilon} + \frac{\pi}{2}\right)(1+|z|)^{2\varepsilon}|z|^{-\varepsilon} \\ &= \left(\frac{2+\pi e\varepsilon}{2e\varepsilon}\right)(|z|^{-1}+2+|z|)^{\varepsilon} \\ &\leq \left(\frac{3}{e\varepsilon}\right)(y^{-1}+2+|z|)^{\varepsilon} \end{aligned}$$

if $\varepsilon \leq 1/3$ and $z = x + iy \in \mathcal{H}$. Choosing $\varepsilon = \frac{3}{\ell}$ yields

$$\begin{aligned} \left|\log \frac{z}{i}\right|^{\ell} &\leq \left(\tfrac{\ell}{e}\right)^{\ell}(y^{-1}+2+|z|)^3 \\ &\leq K\left(\tfrac{\ell}{e}\right)^{\ell}(|z|^4 + y^{-4}) \end{aligned} \tag{3.3}$$

for $\ell \geq 9$ and every $z \in \mathcal{H}$. Here K is a constant independent of z. If we set $\alpha = \max_{1\leq j\leq M}(|\alpha_j|)$, then $\left|\left(\frac{z}{i}\right)^{-\alpha_j}\right| = |z|^{-\operatorname{Re}(\alpha_j)} e^{-\operatorname{Im}(\alpha_j)\arg(z/i)} \leq |z|^{-\operatorname{Re}(\alpha_j)} e^{\frac{\pi}{2}\alpha}$. By (i), we have that $|z|^{-\operatorname{Re}(\alpha_j)} \leq (|z|^{\alpha} + y^{-\alpha})$. Thus

$$\left|(z/i)^{-\alpha_j}\right| \leq e^{\frac{\pi}{2}\alpha}(|z|^{\alpha} + y^{-\alpha}). \tag{3.4}$$

Using the Weierstrass M-Test together with the estimates (3.3) and (3.4), we can easily prove the following proposition.

Proposition 3.1. *Let* $q(z) = \sum_{j=1}^{M} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{\infty} \delta(j,\ell)\left(\log \frac{z}{i}\right)^{\ell}$. *Suppose that*

(i) $|\operatorname{Re}(\alpha_j)| \leq \alpha$ *for all* j, *where* $\alpha \in \mathbb{R}^+$, *(ii)* $\sum_{\ell=1}^{\infty} \left(\frac{\ell}{e}\right)^{\ell} |\delta(j,\ell)| < \infty$ *for* $j = 1, \ldots, M$.

Then $q(z)$ *converges absolutely for all* $z \in \mathcal{H}$*, uniformly on compact subsets of* $\mathcal{H}$*, and* $q \in \mathcal{P}$*. Furthermore, the series for* $q(-1/z)$ *also converges absolutely and uniformly on compact subsets of* $\mathcal{H}$*.*

Remark. Because there are only finitely many α_j, there is always some α for which (i) holds. However, in the next chapter (i) will be a necessary assumption. We include (i) here to draw parallels between the two cases. The difference in convergence will be in the assumptions made on the coefficients $\delta(j,\ell)$.

Proof. From (3.3) and (3.4) we obtain the estimate

$$\left|\sum_{j=1}^{M}(\tfrac{z}{i})^{-\alpha_j}\sum_{\ell=9}^{\infty}\delta(j,\ell)(\log\tfrac{z}{i})^{\ell}\right| \leq \sum_{j=1}^{M}\left|(\tfrac{z}{i})^{-\alpha_j}\right|\sum_{\ell=9}^{\infty}|\delta(j,\ell)||\log\tfrac{z}{i}|^{\ell}$$

$$\leq Ke^{\frac{\pi}{2}\alpha}(|z|^{\alpha}+y^{-\alpha})(|z|^4+y^{-4})\sum_{j=1}^{M}\sum_{\ell=9}^{\infty}(\tfrac{\ell}{e})^{\ell}|\delta(j,\ell)|.$$

For $0 \leq \ell < 9$, we have $|\log\frac{z}{i}|^{\ell} \leq 2^{\ell}(|z|+y^{-1})^{\ell}$. Thus

$$|q(z)| \leq e^{\frac{\pi}{2}\alpha}(|z|^{\alpha}+y^{-\alpha})\left(\sum_{j=1}^{M}\sum_{\ell=0}^{8}|\delta(j,\ell)|2^{\ell}(|z|+y^{-1})^{\ell}\right.$$

$$\left.+K(|z|^4+y^{-4})\sum_{j=1}^{M}\sum_{\ell=9}^{\infty}(\tfrac{\ell}{e})^{\ell}|\delta(j,\ell)|\right).$$

By (ii), the last sum converges. Thus, by the Weierstrass M-Test, the series defining $q(z)$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$. Furthermore, this estimate also shows us that $q \in \mathcal{P}$.

For $q(-1/z)$, we can easily see that $\log\left(\frac{-1}{iz}\right) = \log\left(\frac{i}{z}\right) = -\log\left(\frac{z}{i}\right)$ whenever $z \in \mathcal{H}$. This implies that

$$\left|\log\left(\tfrac{-1}{iz}\right)\right|^{\ell} \leq K\left(\tfrac{\ell}{e}\right)^{\ell}(|z|^4+y^{-4}) \tag{3.5}$$

for any $\ell \geq 9$. Also, $\left|\left(\frac{-1}{iz}\right)^{-\alpha_j}\right| \leq |z|^{\mathrm{Re}(\alpha_j)}e^{\frac{\pi}{2}\alpha}$ and $|z|^{\mathrm{Re}(\alpha_j)} \leq (|z|^{\alpha}+y^{-\alpha})$. Thus

$$\left|(-1/iz)^{-\alpha_j}\right| \leq e^{\frac{\pi}{2}\alpha}(|z|^{\alpha}+y^{-\alpha}). \tag{3.6}$$

Just as for $q(z)$, the estimates (3.5) and (3.6) together imply that $q(^{-1}/_{z})$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$ by the Weierstrass M-Test. □

Finally, if $q(z)$ is a period function under the inversion, T, for some function $f(z)$, then $q(z)$ satisfies $q|_{\upsilon}^{k}T + q = 0$.

Proposition 3.2. *Suppose that $f(z)$ satisfies the transformation law*

$$z^{-k}f(^{-1}/_{z}) = \upsilon(T)f(z) + q(z) \tag{3.7}$$

for some function $q(z)$. Then $q(z)$ satisfies the equation

$$z^{-k}q(^{-1}/_{z}) + \upsilon(T)q(z) = 0. \tag{3.8}$$

Proof. The transformation law (3.7) for f implies that $\upsilon(T)f(^{-1}/_{z}) = (^{-1}/_{z})^{-k}f(z) - q(^{-1}/_{z})$. Plugging this back into (3.7) and rearranging slightly yields the equation

$$f(z)\left(1 - \upsilon(T)^2(^{-1}/_{z})^k z^k\right) = \upsilon(T)(^{-1}/_{z})^k z^k q(z) + (^{-1}/_{z})^k q(^{-1}/_{z}). \tag{3.9}$$

Now the expression $(^{-1}/_{z})^k z^k$ is actually constant. For $z = iy$ for $y > 0$, we see that

$$(^{-1}/_{z})^k z^k = (^{i}/_{y})^k (iy)^k = i^k \tfrac{1}{y^k} i^k y^k = (i^k)^2.$$

By the identity theorem, this extends to the entire upper half-plane. Hence equation (3.9) becomes

$$f(z)\left(1 - (i^k\upsilon(T))^2\right) = \upsilon(T)(^{-1}/_{z})^k z^k q(z) + (^{-1}/_{z})^k q(^{-1}/_{z}). \tag{3.10}$$

But υ is a multiplier system in weight k, so $\upsilon(T) = \pm i^{-k}$. This means that $1 - (i^k\upsilon(T))^2 = 0$, and (3.8) follows. □

3.2 Analytic Results

Here we will state and prove (or give a reference for) the analytic results required for the proof of Theorem 3.8. We will begin with a rather useful lemma about integration. It is easy to prove using induction and integration by parts.

Lemma 3.3. *$\int_1^\infty y^w (\log y)^\ell dy = (-1)^{\ell+1}\ell!(w+1)^{-\ell-1}$ for $w \in \mathbb{C}$ with $\mathrm{Re}(w) < -1$ and $\ell \in \mathbb{N} \cup \{0\}$.*

We shall also require Stirling's formula and a formulation of the Phragmén-Lindelöf theorem for 'lacunary' vertical strips.

Theorem 3.4 (Stirling's Formula I)**.** *For any $\delta > 0$,*

$$\log\Gamma(s) = \left(s - \tfrac{1}{2}\right)\log s - s + \tfrac{1}{2}\log(2\pi) + \mathcal{O}\left(\tfrac{1}{|s|}\right)$$

in $|\arg s| \le \pi - \delta$, where the implied constant depends only upon δ.

For this formula, we refer the reader to [11]. One can use this theorem to easily prove

Corollary 3.5 (Stirling's Formula II)**.** *For fixed real σ,*

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} e^{-\pi|t|/2}, \tag{3.11}$$

as $|t| \to \infty$.

Thus for any $s = \sigma + it$ with $|t| \ge 1$ in the strip $S(a, b) = \{\sigma + it : a < \sigma < b\}$, we have

$$|\Gamma(s)| \le K|t|^\rho e^{-\pi|t|/2}, \tag{3.12}$$

Here K depends only upon a and b while ρ depends only upon a.

We next include two variations on the Phragmén-Lindelöf theorem. As these are not standard variations, we provide proofs of these theorems.

Theorem 3.6. *Let $S(a, b)$ be the vertical strip $\{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$, and let $\Omega \subseteq S(a, b)$ be any open subset of $S(a, b)$. Suppose that $f(z)$ is analytic on Ω and continuous on $\partial\Omega$ such that*

(1) $|f(z)| \le M$ for $z \in \partial\Omega$,

(2) $f(z) = \mathcal{O}_{\Omega,\theta}\big(\exp(e^{\theta\pi|z|/(b-a)})\big)$,

where $\theta < 1$ and the implied constant is independent of z (but may depend upon Ω and θ). Then $|f(z)| \le M$ for all $z \in \Omega$.

Proof. First, we claim that it is enough to prove the theorem in the special case $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$. To see this, assume the theorem in this case, and let f be the given function on $\Omega \subset S(a,b)$. Then let $\varphi(z) = \frac{\pi}{b-a}\left(z - \frac{a+b}{2}\right)$ be the Möbius transformation that scales and shifts $S(a,b)$ onto $S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We then want to apply the theorem in the special case to the function $\hat{f}(z) = f(\varphi(z))$ on $\varphi(\Omega) \subset S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $\varphi(\partial\Omega) = \partial\varphi(\Omega)$, $|\hat{f}(z)| \leq M$ for $z \in \partial\varphi(\Omega)$. To apply the theorem, we also need to check that $\hat{f}$ has the correct growth condition (2) in $\varphi(\Omega) \subseteq S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now

$$e^{\theta\pi|\varphi^{-1}(z)|/(b-a)} \leq K(\Omega,\theta)e^{\theta|z|}$$

for a constant $K(\Omega,\theta)$ that depends only on Ω and θ. Thus for any $z \in \varphi(\Omega)$,

$$\hat{f}(z) = \mathcal{O}_{\Omega,\theta}\left(\exp(e^{\theta|z|})\right).$$

We can then apply the theorem to $\hat{f}$ to get $|\hat{f}(z)| \leq M$ for all $z \in \varphi(\Omega)$. This implies that $|\hat{f}(\varphi^{-1}(z))| = |f(z)| \leq M$ for all $z \in \Omega$.

Now assume that $b = -a = \pi/2$. Consider the function

$$g(z) = f(z)\exp(-\rho e^{-i\kappa z})$$

where $\rho > 0$ and $\max(0,\theta) < \kappa < 1$. To study the growth of g in $\overline{\Omega}$, set $w(z) = \exp(-\rho e^{-i\kappa z})$. Then

$$|w(z)| = \exp(-\rho(\cos\kappa x)e^{\kappa y}) \leq \exp(-\rho(\cos\frac{\kappa\pi}{2})e^{\kappa y})$$

because $|x| = |\operatorname{Re}(z)| \leq \pi/2$ and $0 < \kappa < 1$. For g, we have

$$g(z) = \mathcal{O}\left(\exp\left(e^{\theta|z|} - \rho e^{\kappa y}\cos(\kappa\pi/2)\right)\right)$$

uniformly for $z \in \Omega$. Now, for the exponent in this estimate, we can see that

$$e^{\theta|z|} - \rho e^{\kappa y}\cos(\kappa\pi/2) \leq e^{\theta y}\left(e^{\pi/2} - \rho\cos(\kappa\pi/2)e^{(\kappa-\theta)y}\right).$$

Since $\kappa > \theta$, the right hand side of this inequality approaches $-\infty$ as $y \to +\infty$. Thus $|g(z)| \to 0$ as $y \to +\infty$, and so there exists some $y_M > 0$ such that $|g(z)| \leq M$ for all $z = x + iy$ with $y \geq y_M$ and $|x| \leq \pi/2$.

Let $z_0 = x_0 + y_0$ be any point in Ω. If $y_0 \geq y_M$, then $|g(z_0)| \leq M$ by the work above. If $y_0 < y_M$, set $\Omega_M = \{z \in \overline{\Omega} : |\operatorname{Im}(z)| \leq y_M\}$. Then $|g(z)| \leq M$ for all $z \in \overline{\Omega} \cap \{|\operatorname{Im}(z)| = y_M\}$ by our choice of y_M. For the rest of the region Ω_M, note that $|w(z)| \leq 1$ for all $z \in \overline{\Omega}$, and thus $|g(z)| = |f(z)||w(z)| \leq M$ for all $z \in \partial\Omega_M$ by our assumption on f. We can now apply the maximum modulus principle to see that $|g(z)| \leq M$ for all $z \in \Omega_M$, and in particular, $|g(z_0)| \leq M$. This proves that $|g(z_0)| \leq M$ for any $z_0 \in \Omega$.

We now return to f and can easily see that

$$|f(z_0)| \leq |g(z_0)||w(z_0)|^{-1} \leq M \exp\left(\rho(\cos(\kappa x_0)e^{\kappa y_0})\right).$$

Now let $\rho \to 0^+$ to get that $|f(z_0)| \leq M$ as desired. □

We can use this theorem to prove a version that allows for moderate growth along the boundary of Ω. We shall also restrict ourselves to subsets of a 'lacunary' vertical strip.

Corollary 3.7. *Let Ω be an open subset of the lacunary vertical strip $S_\eta(a,b) = \{z = x + iy \in \mathbb{C} : a < x < b, |y| > \eta\}$ for some $\eta > 0$. Suppose that $f(z)$ is analytic on Ω and continuous on $\partial\Omega$. Also suppose that*

(1) $f(z) = \mathcal{O}(|y|^\alpha)$ on $\partial\Omega$ for some $\alpha \in \mathbb{R}^+$,

(2) $f(z) = \mathcal{O}_{\Omega,\theta}\left(\exp(e^{\theta\pi|z|/(b-a)})\right)$,

where $\theta < 1$ and the implied constant is independent of z. Then $f(z) = \mathcal{O}(|y|^\alpha)$, uniformly in Ω.

Proof. Write $\Omega = \Omega^+ \cup \Omega^-$, where $\Omega^+ = \{z \in \Omega : \operatorname{Im}(z) > \eta\}$ and $\Omega^- = \{z \in \Omega : \operatorname{Im}(z) < -\eta\}$, and $U = \{z \in \Omega : -\eta \leq \operatorname{Im}(z) \leq \eta\}$. We shall prove the result for Ω^+ and Ω^-.

For Ω^-, it is sufficient to prove the result with Ω replaced by Ω^+. To see this, define $\hat{f} : \Omega^+ \to \mathbb{C}$ by $\hat{f}(z) = \overline{f(\overline{z})}$. The function $\hat{f}$ is analytic in Ω^+ and satisfies (1) and (2) since f satisfies these conditions. Thus we can invoke the result for Ω^+ and get $\hat{f}(z) = \mathcal{O}(|y|^\alpha)$ uniformly in Ω^+. But this immediately implies that $f(z) = \mathcal{O}(|y|^\alpha)$, uniformly in Ω^- by the definition of $\hat{f}$. Thus we only need to prove the corollary with Ω replaced by Ω^+.

In this case, let $\psi(z) = (-iz)^{-\alpha} = e^{-\alpha \log(-iz)}$ and $\tilde{f}(z) = f(z)\psi(z)$. Then

$$|\psi(z)| = e^{-Re(\alpha \log(-iz))} = e^{-\alpha \log|-iz|} = |-iz|^{-\alpha} = |y - ix|^{-\alpha}.$$

Since $a \leq x \leq b$ for $z \in \overline{\Omega^+}$, we also have that

$$|y - ix|^{-\alpha} = \mathcal{O}(y^{-\alpha})$$

as $y \to +\infty$. Here the implied constant is independent of z. Thus $\psi(z) = \mathcal{O}\left(y^{-\alpha}\right)$ for all $z \in \overline{\Omega^+}$ with the implied constant independent of z.

This allows us to estimate $\tilde{f}(z) = f(z)\psi(z)$ on the boundary of Ω^+. Using this last estimate and the hypothesis (i), we have

$$\tilde{f}(z) = \mathcal{O}\left(y^{\alpha} \cdot y^{-\alpha}\right) = \mathcal{O}(1)$$

for all $z \in \partial\Omega^+$. That is to say, there is a positive constant K such that $|\tilde{f}(z)| \leq K$ for every $z \in \partial\Omega^+$ (K depends on a, b, and η, but not on z). Finally, to apply Theorem 3.6 we also require a growth condition on $\tilde{f}$ inside Ω^+. But since $y^{-\alpha} \to 0$ as $y \to +\infty$, we know that $\psi(z) = \mathcal{O}(1)$ in Ω^+. Combining this estimate with the growth condition (2) on f, we have

$$\tilde{f}(z) = \mathcal{O}_{\Omega,\theta}\left(\exp(e^{\theta\pi|z|/(b-a)}) \cdot 1\right) = \mathcal{O}_{\Omega,\theta}\left(\exp(e^{\theta\pi|z|/(b-a)})\right).$$

Then we apply Theorem 3.6 to $\tilde{f}$ to conclude that

$$|f(z)||\psi(z)| = |\tilde{f}(z)| \leq K, \quad \text{for } z \in \Omega^+.$$

Because $\psi(z) \neq 0$ in Ω^+, this implies that $|f(z)| \leq K/|\psi(z)|$ for $z \in \Omega^+$. The last step is to get an upper bound on $|\psi(z)|$.

From our earlier calculation of $|\psi(z)|$, we have that

$$|\psi(z)| = |y - ix|^{-\alpha} = y^{-\alpha}\left(\sqrt{1 + \tfrac{x^2}{y^2}}\right)^{-\alpha}.$$

But since $a \leq x \leq b$ and $\alpha > 0$,

$$\left(\sqrt{1 + \tfrac{x^2}{y^2}}\right)^{-\alpha} \geq \delta > 0$$

uniformly in $\overline{\Omega^+}$ for some δ independent of z. Thus $|\psi(z)| \geq \delta y^{-\alpha}$ and $1/|\psi(z)| = \mathcal{O}(y^{\alpha})$. In conjunction with the inequality $|f(z)| \leq K/|\psi(z)|$, this implies that $f(z) = \mathcal{O}(y^{\alpha})$ uniformly in Ω^+ as $y \to +\infty$. $\square$

3.3 The Correspondence Theorem

In this section, we shall prove the correspondence theorem for automorphic integrals with infinite log-polynomial period functions. Throughout we will assume that condition (i) of Proposition 3.1 holds. We shall further assume that

(ii^*) $\sum_{j=1}^{M}\sum_{\ell=0}^{\infty} \ell!|\delta(j,\ell)|\varepsilon^{\ell} < \infty$ for every $\varepsilon > 0$.

Note that condition (ii^*) is stronger than condition (ii) from Proposition 3.1, and so any infinite log-polynomial sum of the form (3.1) shall converge absolutely and uniformly on compact subsets of $\mathcal{H}$. We shall also assume that υ is a multiplier system in weight k on $G(\lambda)$ with $\upsilon(S_\lambda) = 1$.

Theorem 3.8. *Let* $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$ *be holomorphic in* $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\gamma)$ *for some* $\gamma > 0$. *Let* $q(z) = \sum_{j=1}^{M} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{\infty} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell}$ *and define the functions*

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\Phi_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_f(s),$$

$$Q(s) = \overline{\upsilon}(T) \sum_{j=1}^{M}\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} + a_0\left(\frac{i^k\upsilon(T)}{s-k} - \frac{1}{s}\right).$$

Then the following are equivalent:

(A) $f(z)$ *satisfies the transformation law*

$$z^{-k} f(-1/z) = \upsilon(T) f(z) + q(z). \tag{3.13}$$

(B) $\Phi_f(s) - Q(s)$ *has an analytic continuation to the entire* s*-plane that is bounded in vertical strips, and* $\Phi_f(k-s) = i^k \upsilon(T)\, \Phi_f(s)$.

Proof. $(A) \implies (B)$. Consider the Mellin transform, Φ_f, of $f(iy) - a_0$:

$$\begin{aligned}
\Phi_f(s) &= \int_0^\infty (f(iy) - a_0)\, y^s \frac{dy}{y} \\
&= \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{-2\pi n y/\lambda} \right) y^s \frac{dy}{y} \\
&= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n y/\lambda} y^{s-1}\, dy \\
&= \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} \\
&= \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \varphi_f(s),
\end{aligned}$$

provided that $\mathrm{Re}(s) > \gamma + 1$. Now we can also write

$$\begin{aligned}
\Phi_f(s) &= \int_0^\infty (f(iy) - a_0)\, y^s \frac{dy}{y} \\
&= \int_1^\infty (f(iy) - a_0)\, y^{s-1}\, dy + \int_0^1 (f(iy) - a_0)\, y^{s-1}\, dy \\
&= \int_1^\infty (f(iy) - a_0)\, y^{s-1}\, dy + \int_1^\infty (f(i/y) - a_0)\, y^{-s-1}\, dy \\
&= I + II.
\end{aligned}$$

Now by the transformation law (3.13),

$$\begin{aligned}
II = i^k \upsilon(T) \int_1^\infty (f(iy) - a_0) y^{k-s-1}\, dy + i^k \upsilon(T) \int_1^\infty a_0 y^{k-s-1}\, dy \\
- \int_1^\infty a_0 y^{-s-1}\, dy + i^k \int_1^\infty q(iy) y^{k-s-1}\, dy.
\end{aligned}$$

Note that each of these integrals converges for $\mathrm{Re}(s) > \max(\gamma+1, 4+\alpha+|k|+1)$. Using Lemma 3.3, we can explicitly evaluate the two middle integrals to get

$$\begin{aligned}
II = i^k \upsilon(T) \int_1^\infty (f(iy) - a_0) y^{k-s-1}\, dy \\
+ a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right) + i^k \int_1^\infty q(iy) y^{k-s-1}\, dy.
\end{aligned}$$

This gives us that $\Phi_f(s) = E(s) + R(s) + L(s)$, where

$$\begin{aligned} E(s) &= \int_1^\infty (f(iy) - a_0)\, y^{s-1}\, dy + i^k \upsilon(T) \int_1^\infty (f(iy) - a_0)\, y^{k-s-1}\, dy \\ R(s) &= a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right) \\ L(s) &= i^k \int_1^\infty q(iy) y^{k-s-1}\, dy. \end{aligned}$$

The function $E(s)$ is easily seen to be entire, and since $\left(i^k \upsilon(T)\right)^2 = 1$, we also have that $E(k-s) = i^k \upsilon(T)\, E(s)$. Furthermore, $E(s)$ is bounded in vertical strips. Next, $R(s)$ is meromorphic in the entire s-plane, and of course $R(k-s) = i^k \upsilon(T)\, R(s)$.

Next, we apply the Lebesgue Dominated Convergence Theorem to integrate $q(iy)y^{k-s-1}$ term by term so that we have an exact expression for $L(s)$. To justify the use of the Lebesgue Dominated Convergence Theorem, note that since $y \geq 1$, we have $\log y \leq \frac{\ell}{e} y^{1/\ell}$ for any $\ell \geq 1$. Then $|\log y|^\ell \leq \ell! y$ for $\ell \geq 0$. Hence

$$\begin{aligned} |q(iy)y^{k-s-1}| &\leq \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \left| y^{-\alpha_j} \right| |\delta(j,\ell)| |\log y|^\ell\, y^{k-\mathrm{Re}(s)-1} \\ &\leq \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} y^{\alpha} |\delta(j,\ell)|\, \ell! y\, y^{k-\mathrm{Re}(s)-1} \\ &= \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \ell! |\delta(j,\ell)|\, y^{k-\mathrm{Re}(s)+\alpha}. \end{aligned}$$

Thus the function $g(y) = \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \ell! |\delta(j,\ell)|\, y^{k-\mathrm{Re}(s)+\alpha}$ dominates $|q(iy)y^{k-s-1}|$. By the Monotone Convergence Theorem, we can integrate g term by term. From Lemma 3.3, we see that the integral of any term is

$$\begin{aligned} \int_1^\infty y^{k-\mathrm{Re}(s)+\alpha} \ell! |\delta(j,\ell)|\, dy &= -\ell! |\delta(j,\ell)| (k - \mathrm{Re}(s) + \alpha + 1)^{-1} \\ &= \frac{\ell! |\delta(j,\ell)|}{\mathrm{Re}(s) - \alpha - k - 1} \end{aligned}$$

whenever $\mathrm{Re}(s) > 1 + \alpha + k$. So

$$\int_1^\infty g(y)\,dy = \sum_{j=1}^M \sum_{\ell=0}^\infty \frac{\ell!|\delta(j,\ell)|}{\mathrm{Re}(s) - k - \alpha - 1}.$$

This sum converges for $\mathrm{Re}(s) > |k| + \alpha + 1$ by assumption (ii^*). Thus by the Lebesgue Dominated Convergence Theorem, we can integrate $q(iy)y^{k-s-1}$ term by term. The integral of a single term is

$$\int_1^\infty y^{-\alpha_j+k-s-1}\delta(j,\ell)(\log y)^\ell\,dy = (-1)^{\ell+1}\ell!\delta(j,\ell)(k-s-\alpha_j)^{\ell+1}$$

by Lemma 3.3. Thus

$$i^k \int_1^\infty q(iy)y^{k-s-1}\,dy = i^k \sum_{j=1}^M \sum_{\ell=0}^\infty \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(k-s-\alpha_j)^{\ell+1}}. \tag{3.14}$$

This sum is actually holomorphic on $\mathbb{C} \setminus \{k - \alpha_j\}$: let V be a compact subset of $\mathbb{C} \setminus \{k - \alpha_j\}$. Then there exists some $\varepsilon > 0$ such that $|s - (k - \alpha_j)| \geq \varepsilon$ for all j and $s \in V$. This means that

$$\sum_{j=1}^M \sum_{\ell=0}^\infty \left| \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(k-s-\alpha_j)^{\ell+1}} \right| \leq \sum_{j=1}^M \sum_{\ell=0}^\infty \ell!|\delta(j,\ell)|\varepsilon^{-\ell-1}$$

for every $s \in V$. But this sum converges by (ii^*), so the sum converges absolutely and uniformly on V by the Weierstrass M-Test. Thus the sum on the right-hand side of (3.14) is holomorphic in $\mathbb{C} \setminus \{k - \alpha_j\}$.

Because $f(z)$ satisfies (3.13), we know by Proposition 3.2 that $q(z)$ satisfies the transformation law (3.8). That is,

$$\begin{aligned} q(iy) &= -\overline{\upsilon}(T)(iy)^{-k} q({}^{i}\!/\!_{y}) \\ &= -\overline{\upsilon}(T) i^{-k} \sum_{j=1}^M \sum_{\ell=0}^\infty y^{\alpha_j - k}\delta(j,\ell)(-\log y)^\ell. \end{aligned} \tag{3.15}$$

The function $h(y) = \sum_{j=1}^M \sum_{\ell=0}^\infty y^{-\mathrm{Re}(s)+\alpha}\ell!|\delta(j,\ell)| = y^k g(y)$ dominates $q(iy)y^{k-s-1}$ and is integrable for $\mathrm{Re}(s)$ large. Applying, then, the Lebesgue Dominated

Convergence Theorem to integrate the sum in (3.15) term by term, we arrive at another expression for the integral:

$$i^k \int_1^\infty q(iy) y^{k-s-1}\, dy = \overline{v}(T) \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1} \ell! \delta(j,\ell)}{(s-\alpha_j)^{\ell+1}}. \tag{3.16}$$

This sum converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\{\alpha_j\}$ by (ii^*). (Note that because $q(z)$ is a log-polynomial sum satisfying $q|_v^k T + q = 0$, the sets $\{k - \alpha_j\}$ and $\{\alpha_j\}$ are actually equal. The two expressions for $i^k \int_1^\infty q(iy) y^{k-s-1}\, dy$ are therefore holomorphic on the same domain.) Comparing (3.14) and (3.16), we immediately see that $L(k-s) = i^k v(T)\, L(s)$.

Now by (3.16), $Q(s) = R(s) + L(s)$. Thus $\Phi_f(s) - Q(s) = E(s)$, where the function $E(s)$ was entire and bounded in vertical strips. We also have that each function $E(s)$, $R(s)$, and $L(s)$ satisfies the correct functional equation, so $\Phi_f(s) = E(s) + R(s) + L(s)$ does as well.

$(B) \implies (A)$. We begin with the Cahen-Mellin integral, which expresses e^{-y} as the inverse Mellin transform of $\Gamma(s)$:

$$e^{-y} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) y^{-s} ds$$

for any $d > 0$. We can use this to see that $f(iy) - a_0$ is the inverse Mellin transform of $\Phi_f(s)$. So,

$$\begin{aligned} f(iy) - a_0 &= \sum_{n=1}^{\infty} a_n e^{-2\pi n y/\lambda} \\ &= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) \left(\frac{2\pi n y}{\lambda}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s} y^{-s} ds \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Phi_f(s) y^{-s} ds, \end{aligned}$$

for any $d > \alpha + \gamma + 1$ and $y > 0$. We will choose $d > \alpha + \gamma + 1 + |k|$.

The next step is to move the line of integration from $\mathrm{Re}(s) = d$ to $\mathrm{Re}(s) = -d$. We do so by integrating around a rectangle with vertices $\pm d \pm iT$, applying the residue theorem, and then allowing T to approach infinity. The integrals along the horizontal edges actually approach zero as T goes to infinity. To prove this, we will use Stirling's formula in combination with the Phragmén-Lindelöf theorem.

Let $\Omega = \{s \in \mathbb{C}\colon -d < \mathrm{Re}(s) < d, |\mathrm{Im}(s)| > 1 + \max(|k|, \alpha)\}$. Then for $s \in \Omega$,

$$\begin{aligned} |Q(s)| &\leq \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \frac{\ell! |\delta(j,\ell)|}{|(s-\alpha_j)|^{\ell+1}} + |a_0| \left(\frac{1}{|s-k|} + \frac{1}{|s|} \right) \\ &\leq \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} \ell! |\delta(j,\ell)| + 2|a_0|. \end{aligned}$$

Because $\Phi_f - Q$ is bounded in vertical strips by assumption, this implies that $\Phi_f(s)$ is bounded in the 'lacunary' vertical strip Ω. Of course, $(\frac{2\pi}{\lambda})^s$ is bounded in any vertical strip (and therefore in Ω).

Now by Stirling's formula (3.11),

$$\Gamma^{-1}(\sigma + it) \sim (2\pi)^{-1/2} |t|^{1/2-\sigma} e^{\pi|t|/2}.$$

Thus $\varphi_f(s) = (\frac{2\pi}{\lambda})^s \Gamma^{-1}(s) \Phi_f(s) = \mathcal{O}(|t|^{1/2+d} e^{\pi|t|/2})$ in Ω. The implied constant here is independent of s.

We also need bounds for φ_f on $\partial\Omega$. On the line $\mathrm{Re}(s) = d$, $\varphi_f(s)$ is an absolutely convergent Dirichlet series (by our choice of d) and therefore bounded in $\mathrm{Im}(s)$. For $\mathrm{Re}(s) = -d$, the functional equation for Φ_f implies that

$$\varphi_f(s) = (i^k \upsilon(T))^{-1} \left(\frac{2\pi}{\lambda} \right)^{2s-k} \varphi_f(k-s) \frac{\Gamma(k-s)}{\Gamma(s)}.$$

On $\mathrm{Re}(s) = -d$, $\varphi_f(k-s)$ is again absolutely convergent, and therefore bounded, because $d > 1 + \gamma + \alpha - k$. By Stirling's formula, we have that

$$\frac{\Gamma(k-s)}{\Gamma(s)} = \mathcal{O}(|t|^{k+2d}),$$

and so $\varphi_f(s) = \mathcal{O}(|t|^{k+2d})$ on the line $\mathrm{Re}(s) = -d$. Thus φ_f satisfies the conditions for Corollary 3.7 in Ω, and we can conclude that

$$\varphi_f(s) = \mathcal{O}(|t|^K) \tag{3.17}$$

as $|t| \to \infty$ (here $K > 0$ and the implied constant is independent of $s \in \Omega$). Combining this with (3.12) in Stirling's formula gives the estimate

$$\Phi_f(s) = \mathcal{O}(|t|^{\rho+K} e^{-\pi|t|/2}) \tag{3.18}$$

uniformly in the lacunary vertical strip Ω. Hence the horizontal integrals $\int_{d\pm iT}^{-d\pm iT} \Phi_f(s) y^{-s} ds$ approach zero as $T \to \infty$.

Thus we can conclude that

$$f(iy) - a_0 = \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} \Phi_f(s) y^{-s} ds + \sum \operatorname{Res}(\Phi_f(s) y^{-s}) \tag{3.19}$$

where the sum ranges over all the singularities of Φ_f (i.e. over the α_j and, if $a_0 \neq 0$, over 0, k). To calculate these residues, recall that $\Phi_f(s) - Q(s)$ is entire. This means that Φ_f has a singularity at α_j with principal part $\overline{\upsilon}(T) \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1} \ell! \delta(j,\ell)}{(s-\alpha_j)^{\ell+1}}$. Next we expand y^{-s} about α_j to get

$$y^{-s} = y^{-\alpha_j} \sum_{m=0}^{\infty} (\log y)^m (-1)^m (s-\alpha_j)^m / m!.$$

Then the expansion of $\Phi_f(s) y^{-s}$ at α_j is given by

$$y^{-\alpha_j} \left(\sum_{m=0}^{\infty} \frac{(-\log y)^m}{m!} (s-\alpha_j)^m \right) \left(\overline{\upsilon}(T) \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1} \ell! \delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} \right),$$

and thus the residue is

$$\overline{\upsilon}(T) y^{-\alpha_j} \sum_{\ell=0}^{\infty} (-1)^{\ell+1} \ell! \delta(j,\ell) \frac{(-\log y)^\ell}{\ell!}. \tag{3.20}$$

We can easily calculate the residues at 0 and k: $\operatorname{Res}(\Phi_f(s) y^{-s}; 0) = -a_0$ and $\operatorname{Res}(\Phi_f(s) y^{-s}; k) = y^{-k} a_0 i^k \upsilon(T)$. With this calculation we can rewrite (3.19) as

$$\begin{aligned} f(iy) - a_0 = \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} & \Phi_f(s) y^{-s} ds + a_0 \left(y^{-k} i^k \upsilon(T) - 1 \right) \\ & + \overline{\upsilon}(T) \sum_{j=1}^{M} \sum_{\ell=0}^{\infty} y^{-\alpha_j} (-1)^{\ell+1} \ell! \delta(j,\ell) \frac{(-\log y)^\ell}{\ell!}. \end{aligned} \tag{3.21}$$

We now apply the functional equation for Φ_f to the integral on the right-hand side of (3.21). Following this, we make the substitution $s \to k - s$ to obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty} \Phi_f(s) y^{-s} ds &= \frac{(i^k \upsilon(T))^{-1}}{2\pi i}\int_{-d-i\infty}^{-d+i\infty} \Phi_f(k-s) y^{-s} ds \\ &= \frac{\overline{\upsilon}(T)}{(iy)^k}\frac{1}{2\pi i}\int_{k+d-i\infty}^{k+d+i\infty} \Phi_f(s)\left(\frac{1}{y}\right)^{-s} ds \\ &= \frac{\overline{\upsilon}(T)}{(iy)^k}\left(f(i/y) - a_0\right)\end{aligned}$$

(where the last line is justified by $k + d > \alpha + \gamma + 1$). Then by (3.21), we have

$$\begin{aligned}f(iy) - a_0 = \frac{\overline{\upsilon}(T)}{(iy)^k}(f(i/y) - a_0) + a_0\left(y^{-k} i^k \upsilon(T) - 1\right) \\ + \overline{\upsilon}(T)\sum_{j=1}^{M}\sum_{\ell=0}^{\infty} y^{-\alpha_j}(-1)^{\ell+1}\ell!\delta(j,\ell)\frac{(-\log y)^\ell}{\ell!}\end{aligned}$$

for $y > 0$. Note that the double sum on the right-hand side is exactly $-q(iy)$. Next, we can extend both sides analytically to hold for all $z \in \mathcal{H}$. Hence,

$$f(z) - a_0 = \frac{\overline{\upsilon}(T)}{z^k}(f(-1/z) - a_0) + a_0\left(\left(\frac{z}{i}\right)^{-k} i^k \upsilon(T) - 1\right) - \overline{\upsilon}(T) q(z).$$

A little bit of algebraic rearranging yields the transformation law (3.13). $\square$

CHAPTER 4

Infinitely Many Poles

4.1 Convergence of the Log-Polynomial Sum

In the case of Dirichlet series with infinitely many poles, we are interested in log-polynomial sums of the form

$$q(z) = \sum_{j=1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell}, \tag{4.1}$$

where the $\{\alpha_j\}$ are all distinct complex numbers, $\delta(j,\ell) \in \mathbb{C}$, and $z \in \mathcal{H}$. In this section, we establish sufficient conditions under which these series will converge absolutely in the upper half-plane, uniformly on compact subsets of $\mathcal{H}$, and be in $\mathcal{P}$.

Proposition 4.1. *Let* $q(z) = \sum_{j=1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell}$*. Suppose that*

(i) $|\operatorname{Re}(\alpha_j)| \leq \alpha$ *for all* $j \geq 1$*, where* $\alpha \in \mathbb{R}^+$*,*

(iii) $\sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \left(\frac{\ell}{e}\right)^{\ell} |\delta(j,\ell)| < \infty$*.*

Then $q(z)$ *converges absolutely, uniformly on compact subsets of* $\mathcal{H}$*, and* $q \in \mathcal{P}$*. Furthermore, the series for* $q(-1/z)$ *also converges absolutely and uniformly on compact subsets of* $\mathcal{H}$*.*

We should say a few words about these two assumptions. In the correspondence theorem that we shall prove, the Dirichlet series will have singularities at α_j for every j. Because we want the Dirichlet series to converge in some right half-plane and satisfy the functional equation $\Phi_f(k-s) = i^k \upsilon(T)\Phi_f(s)$, we need to restrict the singularities to a vertical strip. Therefore there is no loss in assuming (i) at the outset. The condition (iii) is analogous to the assumption (ii) that was made in Proposition 3.1 for log-polynomial sums of the form (3.1).

Proof. For $z \in \mathcal{H}$, we know that $\left|\arg\left(\frac{z}{i}\right)\right| < \frac{\pi}{2}$ by our argument convention. Thus $\left|\left(\frac{z}{i}\right)^{-\alpha_j}\right| = \left|\frac{z}{i}\right|^{-\operatorname{Re}(\alpha_j)} e^{\operatorname{Im}(\alpha_j)\arg(z/i)} \leq |z|^{-\operatorname{Re}(\alpha_j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}$ for any $z \in \mathcal{H}$. From (i) we get that $|z|^{-\operatorname{Re}(\alpha_j)} \leq (|z|^{\alpha} + y^{-\alpha})$, so that

$$\left|\left(\tfrac{z}{i}\right)^{-\alpha_j}\right| \leq e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}(|z|^{\alpha} + y^{-\alpha}). \tag{4.2}$$

Recall also the estimate (3.3) on $|\log(z/i)|^{\ell}$:

$$\left|\log \tfrac{z}{i}\right|^{\ell} \leq K\left(\tfrac{\ell}{e}\right)^{\ell}(|z|^4 + y^{-4}).$$

This holds for $\ell \geq 9$ and all $z = x + iy \in \mathcal{H}$. As in the proof of Proposition 3.1, we can use separate estimates for $0 \leq \ell < 9$ and for $9 \leq \ell \leq M(j)$ to get

$$\begin{aligned}
|q(z)| &\leq (|z|^{\alpha} + y^{-\alpha}) \sum_{j=1}^{\infty} \Bigg(\sum_{\ell=0}^{8} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} |\delta(j,\ell)| 2^{\ell}(|z| + y^{-1})^{\ell} \\
&\qquad\qquad + K(|z|^4 + y^{-4}) \sum_{\ell=9}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \left(\tfrac{\ell}{e}\right)^{\ell} |\delta(j,\ell)| \Bigg) \\
&\leq (|z|^{\alpha} + y^{-\alpha}) \Bigg(2^8(|z| + y^{-1})^8 \sum_{j=1}^{\infty}\sum_{\ell=0}^{8} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} |\delta(j,\ell)| \\
&\qquad\qquad + K(|z|^4 + y^{-4}) \sum_{j=1}^{\infty}\sum_{\ell=9}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \left(\tfrac{\ell}{e}\right)^{\ell} |\delta(j,\ell)| \Bigg).
\end{aligned}$$

By (iii), both $\sum_{j=1}^{\infty}\sum_{\ell=0}^{8} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}|\delta(j,\ell)|$ and $\sum_{j=1}^{\infty}\sum_{\ell=9}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}\left(\frac{\ell}{e}\right)^{\ell}|\delta(j,\ell)|$ converge. The Weierstrass M-Test then implies that $q(z)$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$. The estimate also shows that $q \in \mathcal{P}$.

Now $\log\left({}^{-1}\!/_{iz}\right) = -\log\left(\frac{z}{i}\right)$, and so $|({}^{-1}\!/_{iz})^{-\alpha_j}| \leq K\left(\frac{\ell}{e}\right)^{\ell}(|z|^4 + y^{-4})$ for $\ell \geq 9$. We can therefore use the exact same argument and estimates for $q({}^{-1}\!/_{z})$. □

For the correspondence, we will actually require a stronger condition on the coefficients. Under that more stringent assumption, we can give a rather useful explicit estimate for $|q(z)|$. The precise result is

Corollary 4.2. *Let* $q(z) = \sum_{j=1}^{\infty}\left(\frac{z}{i}\right)^{-\alpha_j}\sum_{\ell=0}^{M(j)}\delta(j,\ell)\left(\log\frac{z}{i}\right)^{\ell}$ *and suppose that*

(i) $|\operatorname{Re}(\alpha_j)| \leq \alpha$ *for all* $j \geq 1$*, where* $\alpha \in \mathbb{R}^+$*,*

(iii)* $\sum_{j=1}^{\infty}\sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}\ell!|\delta(j,\ell)|\varepsilon^{\ell} < \infty$ *for every* $\varepsilon > 0$*.*

Then $q \in \mathcal{P}$ *with*

$$|q(z)| \leq K''\left(|z|^{2\max(\alpha,4)} + y^{-2\max(\alpha,4)}\right) \tag{4.3}$$

where $K'' = K'\sum_{j=1}^{\infty}\sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}\ell!|\delta(j,\ell)|$ *for some positive constant* K'*.*

Proof. Because $|\log x| \leq \frac{1}{e\varepsilon}(x^{\varepsilon} + x^{-\varepsilon})$ for positive real x and $(\frac{\ell}{e})^{\ell} \leq \ell!$, there is a positive constant K such that $|\log\frac{z}{i}|^{\ell} \leq K\ell!(|z|^4 + y^{-4})$ for $\ell \geq 0$. Applying this estimate and (4.2) (along with Corollary 2.3), we have

$$\begin{aligned}|q(z)| &\leq K(|z|^{\alpha} + y^{-\alpha})(|z|^4 + y^{-4})\sum_{j=1}^{\infty}\sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}\ell!|\delta(j,\ell)| \\ &\leq K'(|z|^{2\max(\alpha,4)} + y^{-2\max(\alpha,4)})\sum_{j=1}^{\infty}\sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}}\ell!|\delta(j,\ell)|\end{aligned}$$

for the positive constant $K' = 6K$. □

4.2 The Direct Theorem

In this section, we shall prove the direct half of a correspondence theorem for automorphic integrals with infinite log-polynomial period functions. For the rest of the chapter, we will be working under the conditions (*i*) $|\operatorname{Re}(\alpha_j)| \le \alpha$ for all $j \ge 1$, where $\alpha \in \mathbb{R}^+$,

(*iii**) $\sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \varepsilon^\ell < \infty$ for every $\varepsilon > 0$,

(*iv*) $|\alpha_j| \to \infty$ as $j \to \infty$.

Because condition (*iii**) is stronger than (*iii*), any infinite log-polynomial sum of the form (4.1) shall converge absolutely and uniformly on compact subsets of $\mathcal{H}$. We require (*iv*) so that $\{\alpha_j\}$ has no limit point in $\mathbb{C}$. We shall also assume that υ is a multiplier system in weight k on $G(\lambda)$ with $\upsilon(S_\lambda) = 1$. The direct theorem can then be stated as follows.

Theorem 4.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$ *be holomorphic in* $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\gamma)$ *for some* $\gamma > 0$. *Let* $q(z) = \sum_{j=1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^\ell$ *be a log-polynomial sum satisfying* (*i*), (*iii**), *and* (*iv*), *and define the functions*

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\Phi_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \varphi_f(s),$$

$$Q(s) = \overline{\upsilon}(T) \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \frac{(-1)^{\ell+1} \ell! \delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} + a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s}\right).$$

If $f(z)$ *satisfies the transformation law*

$$z^{-k} f(-1/z) = \upsilon(T) f(z) + q(z), \tag{4.4}$$

then $\Phi_f(s) - Q(s)$ *has an analytic continuation to the entire* s*-plane that is bounded in vertical strips, and* $\Phi_f(k-s) = i^k \upsilon(T)\, \Phi_f(s)$.

Proof. Just as in the direct part of Theorem 3.8, we take the Mellin transform of f and get that $\Phi_f(s) = E(s) + R(s) + L(s)$, where

$$E(s) = \int_1^\infty (f(iy) - a_0)\, y^{s-1}\, dy + i^k \upsilon(T) \int_1^\infty (f(iy) - a_0)\, y^{k-s-1}\, dy,$$

$$R(s) = a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s}\right),$$

$$L(s) = i^k \int_1^\infty q(iy) y^{k-s-1}\, dy.$$

The functions $E(s)$ and $R(s)$ are both meromorphic in the entire s-plane and satisfy the correct functional equation.

For $L(s)$, we can write

$$L(s) = i^k \int_1^\infty \left(\sum_{j=1}^\infty y^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell)\, (\log y)^\ell \right) y^{k-s-1}\, dy.$$

By applying the Lebesgue Dominated Convergence Theorem, we will be able to integrate this sum term by term. Since $y \geq 1$, we have that $|\log y| \leq \frac{1}{e\varepsilon} y^\varepsilon$ for any $\varepsilon > 0$. Choosing $\varepsilon = 1/\ell$ yields $|\log y|^\ell \leq \left(\frac{\ell}{e}\right)^\ell y$ for $\ell \geq 1$. Then $|\log y|^\ell \leq \ell!\, y$, and

$$\begin{aligned} \left| \sum_{j=1}^\infty y^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell)\, (\log y)^\ell\, y^{k-s-1} \right| &\leq \sum_{j=1}^\infty \sum_{\ell=0}^{M(j)} y^{\mathrm{Re}(k-s-1-\alpha_j)} |\delta(j,\ell)|\, |\log y|^\ell \\ &\leq \sum_{j=1}^\infty \sum_{\ell=0}^{M(j)} y^{k-\mathrm{Re}(s)-1-\mathrm{Re}(\alpha_j)} |\delta(j,\ell)| \ell!\, y \\ &= \sum_{j=1}^\infty \sum_{\ell=0}^{M(j)} y^{k-\mathrm{Re}(s)-\mathrm{Re}(\alpha_j)} \ell! |\delta(j,\ell)|. \end{aligned}$$

Letting $g(y) = \sum_{j=1}^\infty \sum_{\ell=0}^{M(j)} y^{k-\mathrm{Re}(s)-\mathrm{Re}(\alpha_j)} \ell! |\delta(j,\ell)|$, we have that $g(y)$ dominates $q(iy)y^{k-s-1}$. Using the Monotone Convergence Theorem and Lemma 3.3, it is not hard to check that g is integrable for $\mathrm{Re}(s) > 1 + \alpha + |k|$. Thus we can integrate $q(iy)y^{k-s-1}$ term by term. Doing so gives us the expression

$$L(s) = i^k \sum_{j=1}^\infty \sum_{\ell=0}^{M(j)} \delta(j,\ell)(-1)^{\ell-1} \ell!\, (k-s-\alpha_j)^{-\ell-1}. \tag{4.5}$$

Because $f(z)$ satisfies the transformation law (4.4), we can apply Proposition 3.2 to get that

$$q(iy) = -\overline{v}(T)\, (iy)^{-k}\, q({}^{i}\!/_{y}).$$

Plugging this into the integral for $L(s)$ gives

$$L(s) = i^k \int_1^\infty -\overline{v}(T)\, (iy)^{-k}\, q({}^{i}\!/_{y}) y^{k-s-1}\, dy = -\overline{v}(T) \int_1^\infty q({}^{i}\!/_{y}) y^{-s-1}\, dy.$$

Using almost the exact same calculation as before, we find that the function $h(y) = \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} y^{\mathrm{Re}(\alpha_j)-\mathrm{Re}(s)} \left(\frac{\ell}{e}\right)^{\ell} |\delta(j,\ell)|$ dominates $q(i/y)y^{-s-1}$. This function is integrable, so we can apply the Lebesgue Dominated Converge Theorem to $q(i/y)y^{-s-1}$. Integrating term by term then yields the expression

$$L(s) = \overline{\upsilon}(T) \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \delta(j,\ell)(-1)^{\ell+1}\ell!(s-\alpha_j)^{-\ell-1}. \tag{4.6}$$

Comparing the two expressions we derived for $L(s)$ and using the fact that $(i^k\upsilon(T))^2 = 1$ immediately yields the functional equation $L(k-s) = i^k\upsilon(T)L(s)$.

Altogether, this shows that $\Phi_f(s) - Q(s)$ has an analytic continuation to the entire s-plane that is bounded in vertical strips and that $\Phi_f(k-s) = i^k\upsilon(T)\Phi(s)$. Furthermore, we can write $\Phi_f(s)$ as

$$\begin{aligned}\Phi_f(s) = &\int_1^{\infty} (f(iy) - a_0)\, y^{s-1}\, dy + i^k\upsilon(T) \int_1^{\infty} (f(iy) - a_0)\, y^{k-s-1}\, dy \\ &+ a_0 \left(\frac{i^k\upsilon(T)}{s-k} - \frac{1}{s}\right) + \overline{\upsilon}(T) \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \delta(j,\ell)(-1)^{\ell+1}\ell!(s-\alpha_j)^{-\ell-1}.\end{aligned} \tag{4.7}$$

□

4.3 Approximating Automorphic Integrals

Let $f(z) = \sum_{n=0}^{\infty} a_n e^{\pi i n z}$ be holomorphic in $\mathcal{H}$ with $a_n = \mathcal{O}(n^{\gamma})$. Suppose that $f(z)$ satisfies the transformation law $\overline{\upsilon}(T)z^{-k}f(-1/z) = f(z) + q(z)$ where $q(z) = \sum_{j=1}^{\infty} (\frac{z}{i})^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,l) \left(\log \frac{z}{i}\right)^{\ell}$ is an infinite log-polynomial sum satisfying (i), (iii^*), and (iv). We shall show that there exist automorphic integrals $f_N(z)$ on Γ_ϑ with finite log-polynomial period functions such that $f_N \to f$ as $N \to \infty$ uniformly on compact subsets of $\mathcal{H}$ using the results of §2.2. Because we are relying on the construction from §2.2, we must assume either that $k \geq 0$ and $\upsilon(T) = i^{-k}$ or that $k > 2$ and $\upsilon(T) = -i^{-k}$. The two cases are nearly identical, so we shall assume in this section that $k \geq 0$ and $\upsilon(T) = i^{-k}$.

First note that $q|T+q=0$ by Proposition 3.2. This means that both $-\alpha_j$ and $k-\alpha_j$ appear as exponents in the log-polynomial sum. By rearranging the sum (if necessary) so that $-\alpha_j$ and $k-\alpha_j$ appear consecutively, we can assume that the function

$$r_N(z) = \sum_{j=2N+1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,l) \left(\log \frac{z}{i}\right)^{\ell}$$

satisfies $r_N|T + r_N = 0$ for all N. Because $|\operatorname{Re}(\alpha_j)| \leq \alpha$ for every j, every function $r_N(z)$ satisfies (i) with the same α. Of course, (iv) also holds for r_N. Next, we know that

$$\sum_{j=2N+1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \varepsilon^{\ell} \leq \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \varepsilon^{\ell} < \infty$$

and so (iii^*) holds for each function r_N as well. Thus $r_N \in \mathcal{P}$.

Let $A = B = 2\max(\alpha, 4)$. From (4.3) we know that

$$\begin{aligned} |r_N(z)| &\leq K'' \sum_{j=2N+1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| (|z|^A + y^{-B}) \qquad (4.8) \\ &\leq \left(K'' \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \right) (|z|^A + y^{-B}) \end{aligned}$$

As the tail of a convergent sum, $\sum\limits_{j=2N+1}^{\infty} \sum\limits_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \to 0$ as $N \to \infty$. Thus (4.8) implies that $r_N(z) \to 0$ uniformly on compact subsets of $\mathcal{H}$ as $N \to \infty$. The second estimate in (4.8) is important in that the constants $A = 2\max(\alpha,4)$, $B = 2\max(\alpha,4)$, and $K = K'' \sum\limits_{j=1}^{\infty} \sum\limits_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)|$ are all independent of N.

Next we generate parabolic cocycles $\{(r_N)_M(z) \colon M \in \Gamma_\vartheta\}$ using $(r_N)_{S_2} \equiv 0$ and $(r_N)_T(z) = r_N(z)$ and form the Poincaré series

$$\Psi_N(z) = \sum_{M \in (\Gamma_\vartheta)_\infty \backslash \Gamma_\vartheta} \frac{(r_N)_M(z)}{(cz+d)^m}. \qquad (4.9)$$

By Proposition 2.4, with $\beta = \max(A/2, B + k/2)$, the series for $\Psi_N(z)$ converges for $m > 2\beta + 4$. Note that any such m will work for every N simultaneously because A, B, and k are independent of N. We now fix m to be a large even integer. Proposition 2.4 further tells us that

$$|\Psi_N(z)| \leq K^* K'' \sum_{j=2N+1}^{\infty} \sum_{\ell=0}^{M(j)} e^{|\operatorname{Im}(\alpha_j)|\frac{\pi}{2}} \ell! |\delta(j,\ell)| \left(\frac{1+4|z|^2}{y^2}\right)^{m/2} (|z|^{6\beta+2k} + y^{-6\beta-2k})$$

where K^* is a positive constant depending only on m. This immediately shows that $\Psi_N(z) \to 0$ uniformly on compact subsets of $\mathcal{H}$ as $N \to \infty$.

Define $H_N(z) = -\frac{\Psi_N(z)}{E_{m,1}(z)}$. Because we are using a fixed m for every N, the poles of the H_N are restricted to the same set of points, only finitely many of which can be in the region

$$\widetilde{\mathcal{R}_\vartheta} = (\overline{\mathcal{R}_\vartheta} \cap \mathcal{H}) \setminus (\{z\colon \operatorname{Re}(z) = 1\} \cup \{z\colon |z| = 1, 0 \leq \operatorname{Re}(z) \leq 1\})$$

of Theorem 2.5. Let $G_N(z)$ be the modular form from Theorem 2.5 with principal parts matching those of $H_N(z)$ and let $z_1, \ldots, z_L$ denote the zeros of $E_{m,1}$ in $\widetilde{\mathcal{R}_\vartheta}$. We need to show that $G_N(z)$ approaches zero uniformly on compact subsets of $\widetilde{\mathcal{R}_\vartheta} \setminus \{z_1, \ldots, z_L\}$.

Recall that we constructed $G_N(z)$ by taking a modular function $g_N(z)$ with appropriate principal parts and defining $G_N(z) = \vartheta^{2k}(z) g_N(z)$. It is thus enough to show that $g_N(z) \to 0$ uniformly as $N \to \infty$. But the function g_N was defined as

$$g_N(z) = g_N(z; i\infty) + g_N(z; -1) + g_N(z; i) + \sum_{t=1}^{L} g_N(z; z_t),$$

so it is enough to show that each function above approaches zero uniformly. Since $E_{m,1}$ is nonzero at $i\infty$, $g_N(z; i\infty) \equiv 0$. Now consider a point $z_t \in \widetilde{\mathcal{R}_\vartheta}$ (the points i and -1 will be analogous). Then $H_N(z)$ has an expansion of the form

$$\frac{\alpha_N(-\rho)}{(z - z_t)^\rho} + \ldots + \frac{\alpha_N(-1)}{z - z_t} + \sum_{n=0}^{\infty} \alpha_N(n)(z - z_t)^n$$

at z_t. Here ρ is the order of the zero of $E_{m,1}$ at z_t and hence independent of N. Since $\Psi_N(z_t)$ might be zero, $\alpha_{-\rho}, \ldots, \alpha_{-1}$ may be zero. We shall also need the expansion of $(\varphi_0(z) - \varphi_0(z_t))^{-1}$:

$$(\varphi_0(z) - \varphi_0(z_t))^{-1} = \frac{d_{-1}}{z - z_t} + \sum_{n=0}^{\infty} d_n (z - z_t)^n,$$

where $d_{-1} \neq 0$. Then the function $g_N(z; z_t)$ is given by

$$g_N(z; z_t) = \sum_{j=1}^{\rho} \beta_N(j)(\varphi_0(z) - \varphi_0(z_t))^{-j}$$

for some $\beta_N(j)$ satisfying

$$\begin{pmatrix} d_{-1}^{\rho} & & \\ & \ddots & \\ * & & d_{-1} \end{pmatrix} \begin{pmatrix} \beta_N(\rho) \\ \vdots \\ \beta_N(1) \end{pmatrix} = \begin{pmatrix} \alpha_N(-\rho) \\ \vdots \\ \alpha_N(-1) \end{pmatrix}. \tag{4.10}$$

Because $\Psi_N(z)$ converges to zero on compact subsets of $\mathcal{H}$, the principal part of $H_N(z)$ at z_t approaches zero as N goes to infinity. That is, $\alpha_N(-j) \to 0$ as $N \to \infty$ for $-\rho \leq -j \leq -1$. Since the matrix on the left-hand side of (4.10) is invertible and independent of N, this forces $\beta_N(j) \to 0$ as $N \to \infty$. There are only finitely many $\beta_N(j)$, so the function $g_N(z; z_t)$ approaches zero uniformly on compact subsets of $\widetilde{\mathcal{R}_\vartheta} \setminus \{z_t\}$. The functions $g_N(z; i)$ and $g_N(z; -1)$ also approach zero uniformly by an analogous argument.

Set $F_N(z) = H_N(z) - G_N(z)$. Then we have that $F_N(z)$ is an automorphic integral with period function $r_N(z)$. Furthermore, $F_N(z) \to 0$ uniformly on compact subsets of $\mathcal{H}$ as $N \to \infty$. Thus, the desired automorphic integrals with finite log-polynomial period functions are $f_N(z) = f(z) - F_N(z)$.

4.4 The Converse Theorem

Theorem 4.4. *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n e^{\pi i n z}$ *be holomorphic in* $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\gamma)$ *for some* $\gamma > 0$. *Let* $q(z) = \sum\limits_{j=1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum\limits_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell}$ *be a log-polynomial* *sum satisfying (i), (iii*), and (iv). Additionally assume that*

(v) There exist $\varepsilon > 0$ and a sequence of positive numbers $T_n \to \infty$ such that $|\operatorname{Im}(\alpha_j) - T_n| \geq \varepsilon$ for all n and j.
Define the functions

$$\begin{aligned}
\varphi_f(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n_s} \\
\Phi_f(s) &= \pi^{-s}\Gamma(s)\varphi_f(s) \\
Q(s) &= \overline{\upsilon}(T) \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} + a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right).
\end{aligned}$$

Suppose that $k \geq 0$ and $\upsilon(T) = i^{-k}$ or that $k > 2$ and $\upsilon(T) = -i^{-k}$. If $\Phi_f(s) - Q(s)$ has an analytic continuation to the entire s-plane that is bounded in vertical strips and $\Phi_f(k-s) = i^k \upsilon(T)\Phi_f(s)$, then $f(z)$ satisfies the transformation law

$$z^{-k} f(-1/z) = \upsilon(T) f(z) + q(z). \tag{4.11}$$

Remarks. 1. In the proof of this theorem, we make essential use of the automorphic integrals constructed in the last section. We therefore must restrict ourselves to the case $\lambda = 2$ and restrict the weight k appropriately.

2. The additional assumption (v) requires that the imaginary parts of the α_j have gaps of a uniform size. That is, $\operatorname{Im}(\alpha_j)$ cannot continually get closer and closer together as $\operatorname{Im}(\alpha_j) \to \infty$.

Proof of Theorem 4.4. We essentially follow the proof of (B) implies (A) in Theorem 3.8. Using the Cahen-Mellin integral, we have that

$$f(iy) - a_0 = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Phi_f(s) y^{-s} ds$$

for $d > \alpha + \gamma + 1 + |k|$. We want to move this line of integration from $\operatorname{Re}(s) = d$ to $\operatorname{Re}(s) = -d$. This is accomplished by integrating around a rectangle with vertices $\pm d \pm iT$ and taking the limit as $T \to \infty$. When Φ_f has only finitely many singularities, we can apply Phragmén-Lindelöf and Stirling's formula to show that the integrals along the horizontal paths approach zero as T goes to infinity. This approach fails when there are infinitely many singularities in the strip from $-d$ to d because we cannot get a polynomial bound on φ_f near each singularity. Instead, we shall use the automorphic integrals discussed in the last section to show that the integrals along the horizontal paths approach zero.

Let $f_N(z) = \sum_{n=0}^{\infty} b_n(N) e^{\pi i n z}$ be the automorphic integral with log-polynomial period function $r_N(z) = \sum_{j=2N+1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,l) \left(\log \frac{z}{i}\right)^{\ell}$ (assuming, of course, any necessary rearranging of the α_j) such that $f_N(z) \to 0$ uniformly on compact subsets of $\mathcal{H}$. As we saw in Theorem 4.3, we can write the Mellin transform as $\Phi_{f_N}(s) = E_N(s) + R_N(s) + L_N(s)$, where

$$E_N(s) = \int_1^{\infty} \left(f_N(iy) - b_0(N)\right) y^{s-1}\, dy + i^k \upsilon(T) \int_1^{\infty} (f_N(iy) - b_0(N)) y^{k-s-1}\, dy$$

$$R_N(s) = b_0(N) \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right)$$

$$L_N(s) = \overline{\upsilon}(T) \sum_{j=2N+1}^{\infty} \sum_{\ell=0}^{M(j)} \frac{(-1)^{\ell+1} \ell! \delta(j,\ell)}{(s-\alpha_j)^{\ell+1}}.$$

We next show that $\Phi_{f_N}(s) \to 0$ uniformly on $V = \bigcup_{n=1}^{\infty} \{s : -d \leq \mathrm{Re}(s) \leq d, \mathrm{Im}(s) = T_n\}$ by showing that each function E_N, R_N, L_N converges to zero uniformly on V.

Starting with $L_N(s)$, note that $|s - \alpha_j| \geq \varepsilon$ for every $s \in V$ and every j by (v). Then

$$|L_N(s)| \leq \sum_{j=2N+1}^{\infty} \sum_{\ell=0}^{M(j)} \ell! |\delta(j,\ell)| \varepsilon^{-\ell-1}$$

for every $s \in V$. This is the tail of the convergent series $\sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \ell! |\delta(j,\ell)| \varepsilon^{-\ell}$ and therefore approaches zero.

Next, for $R_N(s)$, we know that $\left| \frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right|$ is bounded in V. So it suffices to show that $b_0(N) \to 0$. This, however, follows immediately from the Cauchy integral formula and the fact that $f_N(z) \to 0$ uniformly on compact subsets of $\mathcal{H}$.

Finally, for $E_N(s)$, if $s \in V$ and $y \geq 1$, we have that $|y^{s-1}| \leq y^{d-1}$ and $|y^{k-s-1}| \leq y^{k+d-1}$. For uniform convergence on V, it then suffices to show that the integrals $\int_1^\infty |f_N(iy) - b_0(N)| y^{d-1}\, dy$ and $\int_1^\infty |f_N(iy) - b_0(N)| y^{k+d-1}\, dy$ approach zero as $N \to \infty$. We do so by applying the Lebesgue Dominated Convergence Theorem and the fact that $|f_N(iy) - b_0(N)| \to 0$ as $N \to \infty$. Since each $f_N \in \mathcal{P}$ and has an exponential expansion at $i\infty$, we know that the coefficients satisfy $b_n(N) = \mathcal{O}(n^{\gamma(N)})$ for some $\gamma(N) > 0$. However, every function f_N satisfies the same inequality $|f_N(z)| \leq K(|z|^A + y^{-B})$ for K, A, B independent of N. We therefore have $b_n(N) = \mathcal{O}(n^\gamma)$, and the implied constant is also independent of N. Thus

$$\begin{aligned} |f_N(iy) - b_0(N)| &\leq e^{-\pi y} \sum_{n=1}^{\infty} K' n^\gamma e^{-\pi(n-1)} \\ &= K^* e^{-\pi y} \end{aligned}$$

for constants K', K^*. Since $\int_1^\infty e^{-\pi y} y^{d-1}\, dy < \infty$, we can apply the Dominated Convergence Theorem as desired. Similarly, since $\int_1^\infty e^{-\pi y} y^{k+d-1}\, dy < \infty$, we can apply the Dominated Convergence Theorem to $\int_1^\infty |f_N(iy) - b_0(N)| y^{k+d-1}\, dy$. This proves that $E_N(s) \to 0$ uniformly in V.

By applying Theorem 4.3 to f_N, we observe that $\Phi_{f-f_N}(s)$ satisfies the conditions for Theorem 3.8. In particular, in the proof of the converse direction, we show that

$$\lim_{n\to\infty} \int_{\gamma_{\pm T_n}} \Phi_{f-f_N}(s) y^{-s} ds = 0$$

for any fixed N. (Here $\gamma_{\pm T_n}$ is the horizontal path from $d \pm iT_n$ to $-d \pm iT_n$.) We note that

$$\lim_{N\to\infty} \int_{\gamma_{\pm T_n}} \Phi_{f-f_N}(s) y^{-s} ds = \int_{\gamma_{\pm T_n}} \Phi_f(s) y^{-s} ds$$

uniformly in n because $\Phi_{f_N}(s) \to 0$ uniformly on V. Thus we can take the limits in n and N in either order and get

$$\begin{aligned}\lim_{n\to\infty}\int_{\gamma_{\pm T_n}}\Phi_f(s)y^{-s}ds &= \lim_{n\to\infty}\lim_{N\to\infty}\int_{\gamma_{\pm T_n}}\Phi_{f-f_N}(s)y^{-s}ds\\ &= \lim_{N\to\infty}\lim_{n\to\infty}\int_{\gamma_{\pm T_n}}\Phi_{f-f_N}(s)y^{-s}ds\\ &= 0.\end{aligned}$$

By the residue theorem, we can then conclude that

$$f(iy)-a_0=\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi_f(s)y^{-s}ds+\sum\operatorname{Res}(\Phi_f(s)y^{-s}), \tag{4.12}$$

where the sum ranges over all the poles of Φ_f. We can easily calculate these residues:

$$\begin{aligned}&\operatorname{Res}(\Phi_f(s)y^{-s};\alpha_j)=\overline{\upsilon}(T)y^{-\alpha_j}\sum_{\ell=0}^{M(j)}(-1)^{\ell+1}\ell!\delta(j,\ell)\frac{(-\log y)^{\ell}}{\ell!},\\ &\operatorname{Res}(\Phi_f(s)y^{-s};0)=-a_0,\\ &\operatorname{Res}(\Phi_f(s)y^{-s};k)=y^{-k}a_0 i^k\upsilon(T).\end{aligned}$$

We can also use the functional equation for $\Phi_f(s)$ and then make the substitution $s\to k-s$ to obtain

$$\int_{-d-i\infty}^{-d+i\infty}\Phi_f(s)y^{-s}ds=\frac{\overline{\upsilon}(T)}{(iy)^k}(f(i/y)-a_0).$$

Substituting the residues and this expression into (4.12) gives us

$$\begin{aligned}f(iy)-a_0=&\frac{\overline{\upsilon}(T)}{(iy)^k}(f(i/y)-a_0)+a_0\left(y^{-k}i^k\upsilon(T)-1\right)\\ &+\overline{\upsilon}(T)\sum_{j=1}^{\infty}\sum_{\ell=0}^{M(j)}y^{-\alpha_j}(-1)^{\ell+1}\ell!\delta(j,\ell)\frac{(-\log y)^{\ell}}{\ell!}\end{aligned}$$

for $y>0$. We next extend both sides analytically to hold for all $z\in\mathcal{H}$ and note that the double sum on the right-hand side is $-q(z)$ to get the equation

$$f(z)-a_0=\frac{\overline{\upsilon}(T)}{z^k}(f(-1/z)-a_0)+a_0\left(\left(\frac{z}{i}\right)^{-k}i^k\upsilon(T)-1\right)-\overline{\upsilon}(T)q(z).$$

This is, after some simplification, the transformation law (4.11). □

This proves a full correspondence in the case $\lambda = 2$ and for suitable weights k. We record this result here.

Corollary 4.5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n e^{\pi i n z}$ *be holomorphic in* $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\gamma)$ *for some* $\gamma > 0$. *Let* $q(z) = \sum_{j=1}^{\infty} \left(\frac{z}{i}\right)^{-\alpha_j} \sum_{\ell=0}^{M(j)} \delta(j,\ell) \left(\log \frac{z}{i}\right)^{\ell}$ *be a log-polynomial sum satisfying (i) and (iii*). Additionally assume that*

(v) There exist $\varepsilon > 0$ *and a sequence of positive numbers* $T_n \to \infty$ *such that* $|\operatorname{Im}(\alpha_j) - T_n| \geq \varepsilon$ *for all* n *and* j.

Define the functions

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n_s}$$

$$\Phi_f(s) = \pi^{-s}\Gamma(s)\varphi_f(s)$$

$$Q(s) = \overline{\upsilon}(T) \sum_{j=1}^{\infty} \sum_{\ell=0}^{M(j)} \frac{(-1)^{\ell+1}\ell!\delta(j,\ell)}{(s-\alpha_j)^{\ell+1}} + a_0 \left(\frac{i^k \upsilon(T)}{s-k} - \frac{1}{s} \right).$$

Suppose that $k \geq 0$ *and* $\upsilon(T) = i^{-k}$ *or that* $k > 2$ *and* $\upsilon(T) = -i^{-k}$. *Then the following are equivalent.*

(A) $f(z)$ *satisfies the transformation law* $z^{-k} f(-1/z) = \upsilon(T) f(z) + q(z)$.

(B) $\Phi_f(s) - Q(s)$ *has an analytic continuation to the entire* s*-plane that is bounded in vertical strips, and* $\Phi_f(k-s) = i^k \upsilon(T)\, \Phi_f(s)$.

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