SUFFICIENT CONDITIONS AND HIGHER ORDER REGULARITY FOR LOCAL MINIMIZERS IN CALCULUS OF VARIATIONS

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ABSTRACT

SUFFICIENT CONDITIONS AND HIGHER ORDER REGULARITY FOR LOCAL MINIMIZERS IN CALCULUS OF VARIATIONS

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Professor Yury Grabovsky, Chair

We establish sufficient conditions for Lipschitz extremals of integral functionals to be strong local minimizers. We also prove a regularity theorem for those extremals that satisfy our sufficient conditions. Our sufficiency theorem has to be compared with the Grabovsky and Mengesha sufficiency result for smooth extremals, in view of the observation by Kristensen and Taheri that their sufficient conditions do not apply to merely Lipschitz extremals. In this thesis we replace the uniform quasiconvexity condition with a new, much stronger condition that works for non-smooth Lipschitz extremals. We also show that those extremals that satisfy our new condition must be more regular than merely Lipschitz.

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DEDICATION

To my family,

Tamrie Bitew and Felegush Bihonegn

Amsalu Wallellign and Emita T. Bitew

with all love and gratitude

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CHAPTER 1

INTRODUCTION

Variational problems, where the unknown is a vector field are important in non-linear elasticity and martensitic phase transitions in materials science. The observed states are modeled as local or global energy minimizers. If we want to understand whether or not a model captures the essential features of the physical behavior of a material, we need to be able to characterize metastable states, or local minima of the energy. If for scalar variational problem a good understanding has been reached, for vectorial variational problems many fundamental questions are still unanswered. Recently Grabovsky and Mengesha [9] established quasiconvexity-based sufficient conditions for smooth extremals (i.e., solutions of the Euler-Lagrange equation). It was shown by a series of counter-examples [11, 17, 19] that even the uniformly convex variational problems cannot be expected to have smooth solutions. For this reason, it is interesting to try to extend the ideas of Grabovsky and Mengesha to general Lipschitz extremals. This is the purpose of the present work.

The extention of sufficient conditions to the more general case of non-smooth extremals is not trivial. This was shown by an example in [14, Corollary 7.3] of a Lipschitz extremal that satisfies all sufficient conditions of [9], yet fails to be a strong local minimizer. Our approach, as that of [9], is based on the Decomposition Theorem, [7, 13, 9] that permits us to represent an arbitrary variation as a non-interacting superposition of a weak variation and a number (possibly a continuum) of "Weierstrass needles." The uniform positivity of second variation prevents any weak variation from decreasing the functional in the non-smooth case, as well as in the smooth case. Stability with respect to "Weierstrass needles," however, no longer reduces to Morrey's quasiconvexity [16]. The new condition implies quasiconvexity almost everywhere, but is much stronger than that. In this dissertation we show that the two types of sufficient conditions: the uniform positivity of second variation and uniform stability with respect to "Weierstrass needles" guarantee that the Lipschitz solutions of the Euler-Lagrange equation with Dirichlet boundary conditions that satisfy them have to be strong local minimizers.

The new sufficient condition is local in nature and reduces to uniform quasiconvexity at all regular points of the extremal. At the singularities, however, the new condition is far more difficult to understand because it strongly depends on the behavior of the extremal at its singular points. In these cases, the detailed analysis of the new condition is beyond the scope of this dissertation. In this connection, our regularity theorem can be considered as the first step toward understanding our new condition. It says that any extremal satisfying our sufficient conditions have to be of class $W^{2,2}_{\text{loc}}(\Omega;\mathbb{R}^m)$, which restricts somewhat the type of singular behavior of the extremal.

It is instructive to compare our regularity theorem to recent results on partial regularity of strong local minimizers [4, 14]. In this dissertation we make more stringent assumptions than the uniform quasiconvexity required for the above mentioned results. In return we get a *global* $W^{2,2}_{\text{loc}}$ regularity with a minimal subsequent technical effort, while Evans [4]and Kristensen and Taheri [14] get partial $C^{1,\alpha}$ regularity on a dense open subset of full measure.

The dissertation is organized as follows. In Chapter 1 we introduce notation and reformulate the problem as in [8, 9]. In Chapter 2 we recap the well-known necessary conditions and derive a new necessary condition for non smooth strong local minimizers. Then we present sufficient conditions for Lipschitz strong local minimizers. In Chapter 3 we prove the sufficiency theorem. Finally, in Chapter 4, we prove a regularity result for strong local minimizers satisfying the conditions in the sufficiency theorem.

Throughout the thesis we will use the following standard system of notations. For a vector $\boldsymbol{A}$, $|\boldsymbol{A}|$ denote the Euclidean norm, and the Frobenius norm $\sqrt{Tr(\boldsymbol{A}\boldsymbol{A}^t)}$ if $\boldsymbol{A}$ is a matrix. For $1 \leq p \leq \infty$, $\|\boldsymbol{f}\|_p$ denote the L^p norm of $|\boldsymbol{f}(\boldsymbol{x})|$. We use the inner product notation $(\boldsymbol{A}, \boldsymbol{B})$ for the dot product if $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors and the Frobenius inner product $(\boldsymbol{A}, \boldsymbol{B}) = Tr(\boldsymbol{A}\boldsymbol{B}^t)$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same shape. We also use indexless subscript notation for derivatives, such as $W_{\boldsymbol{F}}$ or $W_{\boldsymbol{F}\boldsymbol{F}}$ to denote the tensors with components $\partial W/\partial F_{ij}$ and $\partial^2 W/\partial F_{ij}\partial F_{kl}$ respectively.

1.1 Preliminaries

Here we introduce notations and recast the problem in the form introduced in [8]. We will consider integral functionals of the form

$$E(\boldsymbol{y}) = \int_\Omega W(\nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x}, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ and the Lagrangian $W : \mathbb{M} \to \mathbb{R}$ is assumed to be a continuous function. Here $\mathbb{M}$ denote the space of all $m \times d$ matrices. The functional E is defined on the set of admissible functions:

$$\mathcal{A} = \left\{\boldsymbol{y} \ \in W^{1,\infty}(\Omega;\mathbb{R}^m) : \boldsymbol{y}(\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}), \boldsymbol{x} \in \partial\Omega\right\}, \tag{1.2}$$

where $\partial\Omega$ is smooth (i.e., of class C^1), and $\mathbf{g} \in W^{1,\infty}(\partial\Omega;\mathbb{R}^m)$.

Definition 1 *A function $\boldsymbol{y}_0 \in \mathcal{A}$ is called a weak local minimizer of E if for every sequence $\{\boldsymbol{\phi}_n\} \subset W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\boldsymbol{\phi}_n, \nabla\boldsymbol{\phi}_n \to 0$ in $L^\infty(\Omega;\mathbb{R}^m)$, there exists an N such that $E(\boldsymbol{\phi}_n + \boldsymbol{y}_0) - E(\boldsymbol{y}_0) \geq 0$, for all $n \geq N$.*

Definition 2 *A function $\boldsymbol{y}_0 \in \mathcal{A}$ is called a strong local minimizer of E if for every sequence $\{\boldsymbol{\phi}_n\} \subset W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\boldsymbol{\phi}_n \to 0$ in $L^\infty(\Omega;\mathbb{R}^m)$, there exists an N such that $E(\boldsymbol{\phi}_n + \boldsymbol{y}_0) - E(\boldsymbol{y}_0) \geq 0$, for all $n \geq N$.*

In addition to the continuity of $W(\boldsymbol{F})$, we make the following assumption

C1 **:** $W \in C^2(\mathcal{R})$, where $\mathcal{R}$ is a compact set containing an open neighborhood of the effective range $\mathcal{R}_0$ of $\nabla \boldsymbol{y}_0$, where

$$\mathcal{R}_0 = \{\boldsymbol{F}_0 \in \mathbb{M} : |\{\boldsymbol{x} \in \Omega : |\nabla \boldsymbol{y}_0(\boldsymbol{x}) - \boldsymbol{F}_0| < \epsilon\}| > 0, \text{ for every } \epsilon > 0\}.$$

Let the functional increment be defined by

$$\triangle E(\boldsymbol{\phi}_n) = \int_\Omega \{W(\nabla \boldsymbol{y}_0(\boldsymbol{x}) + \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) - W(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\} d\boldsymbol{x}, \tag{1.3}$$

and

$$\delta E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \frac{\triangle E(\boldsymbol{\phi}_n)}{\alpha_n^2}, \ \alpha_n = \|\nabla \boldsymbol{\phi}_n\|_2.$$

If $\boldsymbol{y}_0$ is a strong local minimizer, then $\boldsymbol{y}_0$ solves the Euler-Lagrange equation in weak form

$$\int_\Omega (W_{\boldsymbol{F}}(\nabla \boldsymbol{y}_0(\boldsymbol{x})), \nabla \boldsymbol{\phi}(\boldsymbol{x})) d\boldsymbol{x} = 0, \tag{1.4}$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

As in [8] instead of $\triangle E(\boldsymbol{\phi}_n)$ we consider

$$\triangle' E(\boldsymbol{\phi}_n) = \int_\Omega W^0(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x}, \tag{1.5}$$

where

$$W^0(\boldsymbol{F}_0, \boldsymbol{H}) = W(\boldsymbol{F}_0 + \boldsymbol{H}) - W(\boldsymbol{F}_0) - (W_{\boldsymbol{F}}(\boldsymbol{F}_0), \boldsymbol{H}).$$

Observe that if $\boldsymbol{y}_0$ solves (1.4), then $\triangle E(\boldsymbol{\phi}_n) = \triangle' E(\boldsymbol{\phi}_n)$.

Let

$$U(\boldsymbol{F}_0, \boldsymbol{H}) = \begin{cases} \frac{W^0(\boldsymbol{F}_0, \boldsymbol{H}) - \frac{1}{2}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{H}, \boldsymbol{H})}{|\boldsymbol{H}|^2} & \text{if } \boldsymbol{H} \neq 0 \\ 0 & \text{if } \boldsymbol{H} = 0, \end{cases}$$

where $\mathsf{L}(\boldsymbol{F}_0) = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{F}_0)$. The function $U(\boldsymbol{F}_0, \boldsymbol{H})$ is continuous in $(\boldsymbol{F}_0, \boldsymbol{H})$ space, vanishing at any $(\boldsymbol{F}_0, 0)$. We can now rewrite the modified increment $\triangle' E$ in terms of $U(\boldsymbol{F}_0, \boldsymbol{H})$.

$$\triangle' E(\boldsymbol{\phi}_n) = \int_\Omega U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) |\nabla \boldsymbol{\phi}_n(\boldsymbol{x})|^2 d\boldsymbol{x} + \frac{1}{2} \int_\Omega (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})) \nabla \boldsymbol{\phi}_n(\boldsymbol{x}), \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x}. \tag{1.6}$$

Let

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \lim_{n\to\infty} \frac{\triangle' E(\boldsymbol{\phi}_n)}{\alpha_n^2},$$

If we define

$$\boldsymbol{\psi}_n(\boldsymbol{x}) = \frac{\boldsymbol{\phi}_n(\boldsymbol{x})}{\alpha_n},$$

then

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \lim_{n\to\infty} \int_\Omega \left[U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{\psi}_n) |\nabla \boldsymbol{\psi}_n|^2 + \frac{1}{2} (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})) \nabla \boldsymbol{\psi}_n, \nabla \boldsymbol{\psi}_n) \right] d\boldsymbol{x}. \quad (1.7)$$

If $\boldsymbol{y}_0$ solves the Euler-Lagrange equation, then $\delta' E = \delta E$.

For our convenience we will use the following shorthand notation

$$\mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G}) = \frac{W^0(\boldsymbol{F}_0, \alpha \boldsymbol{G})}{\alpha^2} = U(\boldsymbol{F}_0, \alpha \boldsymbol{G}) |\boldsymbol{G}|^2 + \frac{1}{2} (\mathsf{L}(\boldsymbol{F}_0) \boldsymbol{G}, \boldsymbol{G}). \quad (1.8)$$

Then we can rewrite (1.7) as

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \lim_{n\to\infty} \int_\Omega \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{\psi}_n(\boldsymbol{x})) d\boldsymbol{x}. \quad (1.9)$$

Our goal is to prove that if $\boldsymbol{y}_0$ satisfies our sufficient conditions, then $\delta E(\{\boldsymbol{\phi}_n\})$ will be greater than some positive number, implying that $\boldsymbol{y}_0$ is a strong local minimizer.

CHAPTER 2

NECESSARY AND SUFFICIENT CONDITIONS

2.1 Necessary conditions

In this section we will recap the well-known necessary conditions and derive a new necessary condition for strong local minimizers.

Euler-Lagrange equation

Consider weak variations of the form $\boldsymbol{\varphi}_\epsilon(\boldsymbol{x}) = \epsilon\boldsymbol{\phi}(\boldsymbol{x})$, for $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. The function

$$\gamma(\epsilon) = \int_\Omega [W(\nabla \boldsymbol{y}_0(\boldsymbol{x}) + \epsilon\nabla\boldsymbol{\phi}(\boldsymbol{x})) - W(\nabla \boldsymbol{y}_0(\boldsymbol{x}))]d\boldsymbol{x}, \tag{2.1}$$

has a local minimum at $\epsilon = 0$, since $\boldsymbol{y}_0$ is a strong local minimizer. Therefore $\gamma'(\epsilon) \mid_{\epsilon=0} = 0$.

To find $\gamma'(\epsilon)$ we can differentiate under the integral sign, and we get

$$\int_\Omega (W_{\boldsymbol{F}}(\nabla \boldsymbol{y}_0(\boldsymbol{x})), \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} = 0, \tag{2.2}$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

Non-negativity of second variation

Recall that the function $\gamma(\epsilon)$ defined by (2.1) has a local minimum at $\epsilon = 0$. Therefore, $\gamma''(\epsilon)\mid_{\epsilon=0} \geq 0$.

Therefore, if $\boldsymbol{y}_0 \in \mathcal{A}$ is a strong local minimizer, then

$$\delta^2 E(\boldsymbol{\phi}) = \int_\Omega (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\nabla\boldsymbol{\phi}(\boldsymbol{x}), \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x} \geq 0, \tag{2.3}$$

for all $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

Quasiconvexity condition

Definition 3 *A function $W : \mathbb{M} \to \mathbb{R}$ is called quasiconvex at $\boldsymbol{F} \in \mathbb{M}$ if for every bounded domain D with $|\partial D| = 0$ we have*

$$W(\boldsymbol{F})|D| \leq \int_D W(\boldsymbol{F} + \nabla\boldsymbol{\phi}(\boldsymbol{x}))d\boldsymbol{x}, \text{ for all } \boldsymbol{\phi} \in W_0^{1,\infty}(D;\mathbb{R}^m), \tag{2.4}$$

where $|D|$and $|\partial\Omega|$ denote the Lebesgue measure of the set.

In [12] it was proved that if $\boldsymbol{y}_0 \in W^{1,\infty}$, the quasiconvexity condition is satisfied at $\nabla\boldsymbol{y}_0(\boldsymbol{a})$, for a.e. $\boldsymbol{a} \in \Omega$. If $\boldsymbol{y}_0 \in C^1(\Omega;\mathbb{R}^m)$, then W is quasiconvex at $\nabla\boldsymbol{y}_0(\boldsymbol{a})$, for every $\boldsymbol{a} \in \Omega$, [1].

2.2 Conditions at infinity

In case of strong variations $\boldsymbol{\phi}_n$, we have no control on the size of $\nabla\boldsymbol{\phi}_n$. For this reason we need to impose conditions on the behavior of $W(\boldsymbol{F})$ at infinity.

***C*2 :** Assume that $W(\boldsymbol{F}) \in C^1(\mathbb{M})$ and satisfies

$$|W(\boldsymbol{F})| \leq C(1 + |\boldsymbol{F}|^p), \tag{2.5}$$

$$|W_{\boldsymbol{F}}(\boldsymbol{F})| \leq C(1 + |\boldsymbol{F}|^{p-1}), \tag{2.6}$$

for all $\boldsymbol{F} \in \mathbb{M}$ and some constant $C > 0$.

Lemma 2.1 *There exists a constant* $C(\mathcal{R}) > 0$ *such that*

$$|U(\boldsymbol{F}_0, \boldsymbol{H}_1)|\boldsymbol{H}_1|^2 - U(\boldsymbol{F}_0, \boldsymbol{H}_2)|\boldsymbol{H}_2|^2| \leq C(\mathcal{R})(|\boldsymbol{H}_1| + |\boldsymbol{H}_2| + |\boldsymbol{H}_1|^{p-1} + |\boldsymbol{H}_2|^{p-1})|\boldsymbol{H}_1 - \boldsymbol{H}_2|, \quad (2.7)$$

for all $\boldsymbol{F}_0 \in \mathcal{R}$, $\boldsymbol{H}_1, \boldsymbol{H}_2 \in \mathbb{M}$.

Proof:

Step 1 : We claim that

if $W(\boldsymbol{F})$ satisfies (2.6), then

$$|W(\boldsymbol{F}_1) - W(\boldsymbol{F}_2)| \leq C(1 + |\boldsymbol{F}_1|^{p-1} + |\boldsymbol{F}_2|^{p-1})|\boldsymbol{F}_1 - \boldsymbol{F}_2|, \quad (2.8)$$

for all $\boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathbb{M}$ and some constant $C > 0$. Indeed, let

$\varphi(t) = W(t\boldsymbol{F}_1 + (1-t)\boldsymbol{F}_2)$, for $t \in [0,1]$. Then $\varphi \in C^1([0,1])$. Observe that

$$W(\boldsymbol{F}_1) - W(\boldsymbol{F}_2) = \int_0^1 \varphi'(t)dt = \int_0^1 (W_{\boldsymbol{F}}(t\boldsymbol{F}_1 + (1-t)\boldsymbol{F}_2), \boldsymbol{F}_1 - \boldsymbol{F}_2)dt,$$

and hence by (2.6)

$$|W(\boldsymbol{F}_1) - W(\boldsymbol{F}_2)| \leq C|\boldsymbol{F}_1 - \boldsymbol{F}_2| \int_0^1 (1 + |t\boldsymbol{F}_1 + (1-t)\boldsymbol{F}_2|^{p-1})dt.$$

The function $\boldsymbol{F} \mapsto |\boldsymbol{F}|^{p-1}$ is convex. Therefore,

$$|t\boldsymbol{F}_1 + (1-t)\boldsymbol{F}_2|^{p-1} \leq t|\boldsymbol{F}_1|^{p-1} + (1-t)|\boldsymbol{F}_2|^{p-1},$$

and we obtain (2.8).

Step 2 :

If $|\boldsymbol{H}_1|$ and $|\boldsymbol{H}_2|$ are small, the inequality (2.7) follows from Taylor's expansion. Suppose $|\boldsymbol{H}_1| \geq 1$ or $|\boldsymbol{H}_2| \geq 1$. Then using (2.8), we get

$$\begin{aligned}
&|U(\boldsymbol{F}_0, \boldsymbol{H}_1)|\boldsymbol{H}_1|^2 - U(\boldsymbol{F}_0, \boldsymbol{H}_2)|\boldsymbol{H}_2|^2| \leq |W(\boldsymbol{F}_0 + \boldsymbol{H}_1) - W(\boldsymbol{F}_0 + \boldsymbol{H}_2)| + \\
&|(W_{\boldsymbol{F}}(\boldsymbol{F}_0), \boldsymbol{H}_1 - \boldsymbol{H}_2)| + \frac{1}{2}|(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{H}_1, \boldsymbol{H}_1) - (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{H}_2, \boldsymbol{H}_2)| \leq \\
&C_1(1 + |\boldsymbol{F}_0 + \boldsymbol{H}_1|^{p-1} + |\boldsymbol{F}_0 + \boldsymbol{H}_2|^{p-1})|\boldsymbol{H}_1 - \boldsymbol{H}_2| + \\
&C_2|\boldsymbol{H}_1 - \boldsymbol{H}_2| + C_3(|\boldsymbol{H}_1| + |\boldsymbol{H}_2|)|\boldsymbol{H}_1 - \boldsymbol{H}_2| \leq \\
&C(|\boldsymbol{H}_1| + |\boldsymbol{H}_2| + |\boldsymbol{H}_1|^{p-1} + |\boldsymbol{H}_2|^{p-1})|\boldsymbol{H}_1 - \boldsymbol{H}_2|.
\end{aligned}$$

■

In addition to growth and regularity conditions, we need the following coercivity condition.

***C*3 :** $W(\boldsymbol{F})$ is bounded from below and satisfies

$$W(\boldsymbol{F}) \geq c_1|\boldsymbol{F}|^p - c_2, \tag{2.9}$$

for some constants c_1, $c_2 > 0$.

Lemma 2.2 *If W is bounded from below and satisfies (2.9), then*

$$W^0(\boldsymbol{F}_0, \boldsymbol{H}) \geq k_1(\mathcal{R})|\boldsymbol{H}|^p - k_2(\mathcal{R})|\boldsymbol{H}|^2, \tag{2.10}$$

for all $\boldsymbol{F}_0 \in \mathcal{R}$, $\boldsymbol{H} \in \mathbb{M}$, and some constants $k_1(\mathcal{R})$, $k_2(\mathcal{R}) > 0$.

Proof : In the derivation below all constants depend on $\mathcal{R}$ and W.

There exists C_0, such that for all $|\boldsymbol{H}| \leq 1$,

$$|W^0(\boldsymbol{F}_0 + \boldsymbol{H})| \leq C_0|\boldsymbol{H}|^2, \text{ for all } \boldsymbol{F}_0 \in \mathcal{R}.$$

Therefore,

$$W^0(\boldsymbol{F}_0 + \boldsymbol{H}) \geq -C_0|\boldsymbol{H}|^2 = |\boldsymbol{H}|^2 - (C_0+1)|\boldsymbol{H}|^2 \geq |\boldsymbol{H}|^p - (C_0+1)|\boldsymbol{H}|^2.$$

For all $\boldsymbol{F}_0 \in \mathcal{R}$, and for all $|\boldsymbol{H}| > 1$, we have

$$W^0(\boldsymbol{F}_0 + \boldsymbol{H}) \geq C_1|\boldsymbol{H}|^p - C_2 - C_3|\boldsymbol{H}| \geq C_1|\boldsymbol{H}|^p - (C_2+C_3)|\boldsymbol{H}|^2.$$

So

$$W^0(\boldsymbol{F}_0, \boldsymbol{H}) \geq k_1|\boldsymbol{H}|^p - k_2|\boldsymbol{H}|^2, \tag{2.11}$$

where $k_1 = \min\{1, C_1\}$, and $k_2 = \max\{C_0+1, C_2+C_3\}$. This finishes the proof of Lemma 2.2.

■

2.3 Sufficient conditions for strong local minima

In the following sections, let $B_\Omega(\boldsymbol{a}, r)$ denote $B(\boldsymbol{a}, r) \cap \overline{\Omega}$.

Theorem 2.1 *Suppose*

i) W *satisfies* $\boldsymbol{C1}$, $\boldsymbol{C2}$, *and* $\boldsymbol{C3}$, *for* $p \geq 2$.

ii) $\boldsymbol{y}_0 \in \mathcal{A}$ *solves Euler-Lagrange equation (2.2) and satisfies the conditions: there exists* $\beta > 0$ *such that*

a)

$$\delta^2 E(\boldsymbol{\phi}) \geq \beta \|\nabla \boldsymbol{\phi}\|_2^2, \text{ for all } \boldsymbol{\phi} \in W_0^{1,\infty}(\Omega; \mathbb{R}^m), \tag{2.12}$$

b) For all $\boldsymbol{a} \in \overline{\Omega}$, *there exists* $r(\boldsymbol{a}) > 0$, *so that for all* $\{\boldsymbol{\phi}_n : n \geq 1\} \subset W_0^{1,\infty}(B_\Omega(\boldsymbol{a}, r); \mathbb{R}^m)$, *such that* $\boldsymbol{\phi}_n \to 0$, *uniformly, as* $n \to \infty$, *and* $\phi_n \to 0$ *in* $W_0^{1,2}(B_\Omega(\boldsymbol{a}, r); \mathbb{R}^m)$, *we have*

$$\varliminf_{n\to\infty} \frac{1}{\|\nabla \boldsymbol{\phi}_n\|_2^2} \int_{B_\Omega(\boldsymbol{a},r)} W^0(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x} \geq \beta. \tag{2.13}$$

Then $\delta E(\{\boldsymbol{\phi}_n\}) \geq \beta$, *for any strong variation* $\{\boldsymbol{\phi}_n\} \subset W_0^{1,\infty}(\Omega; \mathbb{R}^m)$. *In particular* $\boldsymbol{y}_0$ *is a strong local minimizer for the functional* $E(\boldsymbol{y})$.

Theorem 2.1 is a corollary of the following theorem, whose proof is given in the next chapter.

Theorem 2.2 *Assume* W *satisfies* $\boldsymbol{C1}$, $\boldsymbol{C2}$, *and* $\boldsymbol{C3}$, *for* $p \geq 2$. *Suppose* $\boldsymbol{y}_0 \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ *satisfies*

a)' $\delta^2 E(\boldsymbol{\phi}) \geq 0$, *for all* $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$,

b)' For all $\boldsymbol{a} \in \overline{\Omega}$, *there exists* $r(\boldsymbol{a}) > 0$, *so that for all* $\{\boldsymbol{\phi}_n : n \geq 1\} \subset W_0^{1,\infty}(B_\Omega(\boldsymbol{a}, r); \mathbb{R}^m)$, *such that* $\boldsymbol{\phi}_n \to 0$, *uniformly, as* $n \to \infty$, *and* $\phi_n \to 0$ *in* $W_0^{1,2}(B_\Omega(\boldsymbol{a}, r); \mathbb{R}^m)$, *we have*

$$\varliminf_{n\to\infty} \frac{1}{\|\nabla \boldsymbol{\phi}_n\|_2^2} \int_{B_\Omega(\boldsymbol{a},r)} W^0(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{\phi}_n(\boldsymbol{x})) d\boldsymbol{x} \geq 0. \tag{2.14}$$

Then $\delta' E(\{\boldsymbol{\phi}_n\}) \geq 0$, *for any strong variation* $\{\boldsymbol{\phi}_n\} \subset W_0^{1,\infty}(B(0,1);\mathbb{R}^m)$.

Note that in Theorem 2.2 we do not assume that $\boldsymbol{y}_0$ solves Euler-Lagrange equation. The uniform positivity of second variation is no longer required and the conclusion is just the non-negativity of $\delta' E(\{\boldsymbol{\phi}_n\})$. The condition (2.13) is a natural strengthening of (2.14), which is clearly necessary for $\boldsymbol{y}_0$ to be a strong local minimizer.

Proof of Theorem 2.1

Let

$$W_\beta(\boldsymbol{F}) = W(\boldsymbol{F}) - \beta|\boldsymbol{F}|^2.$$

If $\boldsymbol{y}_0$ satisfies inequality (2.13), then $W^0_\beta(\boldsymbol{F}_0, \boldsymbol{H}) = W^0(\boldsymbol{F}_0, \boldsymbol{H}) - \beta|\boldsymbol{H}|^2$ satisfies 2.14. In addition, $\delta^2 E_\beta(\boldsymbol{\phi}) \geq 0$, for all $\boldsymbol{\phi} \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

Thus, Theorem 2.2 implies that

$$\delta' E_\beta(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \frac{\int_\Omega W^0_\beta(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{\phi}_n) d\boldsymbol{x}}{\|\nabla\boldsymbol{\phi}_n\|_2^2} \geq 0, \tag{2.15}$$

for every strong variation $\{\boldsymbol{\phi}_n\}$.

But

$$\frac{\int_\Omega W^0_\beta(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{\phi}_n) d\boldsymbol{x}}{\|\nabla\boldsymbol{\phi}_n\|_2^2} = \frac{\int_\Omega W^0(\boldsymbol{F}_{0)}(\boldsymbol{x}), \nabla\boldsymbol{\phi}_n) d\boldsymbol{x}}{\|\nabla\boldsymbol{\phi}_n\|_2^2} - \beta.$$

So

$$\delta E(\{\boldsymbol{\phi}_n\}) = \delta' E_\beta(\{\boldsymbol{\phi}_n\}) + \beta \geq \beta.$$

■

CHAPTER 3

PROOF OF THEOREM 2.2

3.1 Reduction to the problem of $W^{1,p}$-local minima

First, observe that (2.9) implies that a strong variation whose gradients are unbounded in L^p, has the property that

$$\lim_{n\to\infty} \Delta E(\boldsymbol{\phi}_n) = +\infty.$$

Hence, $\delta' E(\{\boldsymbol{\phi}_n\}) \geq 0$. Thus, we may restrict our attention only to variations $\{\boldsymbol{\phi}_n\}$ for which the sequence $\|\nabla\boldsymbol{\phi}_n\|_p$ is bounded. In particular, extracting a subsequence, if necessary, we may assume, without loss of generality that $\boldsymbol{\phi}_n$ converges to zero in the weak topology of $W^{1,p}$. Define

$$\alpha_n = \|\nabla\boldsymbol{\phi}_n\|_2, \text{ and } \beta_n = (2|\Omega|)^{\frac{1}{2}-\frac{1}{p}}\|\nabla\boldsymbol{\phi}_n\|_p. \tag{3.1}$$

Notice that β_n is bounded and $\alpha_n \leq \beta_n$. Hence the sequence α_n is bounded as well. Thus, without loss of generality, $\alpha_n \to \alpha_0 < +\infty$, as $n \to \infty$.

Let us first consider the case, $\alpha_0 > 0$. We have

$$\varliminf_{n\to\infty} \int_\Omega W(\nabla\boldsymbol{y}_0(\boldsymbol{x}) + \nabla\boldsymbol{\phi}_n(\boldsymbol{x}))d\boldsymbol{x} \geq \varliminf_{n\to\infty} \int_\Omega QW(\nabla\boldsymbol{y}_0(\boldsymbol{x}) + \nabla\boldsymbol{\phi}_n(\boldsymbol{x}))d\boldsymbol{x},$$

where $QW(\boldsymbol{F})$ is the quasiconvexification of $W(\boldsymbol{F})$ defined by

$$QW(\boldsymbol{F}) = \inf_{\varphi \in W_0^{1,\infty}(B(0,1);\mathbb{R}^m)} \left\{ \frac{1}{|B(0,1)|} \int_{B(0,1)} W(\boldsymbol{F} + \nabla\varphi(\boldsymbol{x})) \right\}.$$

Then the functional

$$\boldsymbol{\phi} \mapsto \int_\Omega QW(\nabla\boldsymbol{\phi}) d\boldsymbol{x}$$

is $W^{1,p}$ sequentially-weak lower semicontinuous [2], and thus,

$$\varliminf_{n\to\infty} \int_\Omega QW(\nabla\boldsymbol{y}_0(\boldsymbol{x}) + \nabla\boldsymbol{\phi}_n(\boldsymbol{x}))d\boldsymbol{x} \geq \int_\Omega QW(\nabla\boldsymbol{y}_0(\boldsymbol{x}))d\boldsymbol{x}.$$

Finally, the quasiconvexity condition $QW(\nabla\boldsymbol{y}_0(\boldsymbol{x})) = W(\nabla\boldsymbol{y}_0(\boldsymbol{x}))$, for a.e. $\boldsymbol{x} \in \Omega$ (see [12]), implies that

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \frac{1}{\alpha_0^2} \varliminf_{n\to\infty} \int_\Omega \left(W(\nabla\boldsymbol{y}_0(\boldsymbol{x}) + \nabla\boldsymbol{\phi}_n) - W(\nabla\boldsymbol{y}_0(\boldsymbol{x}))\right) d\boldsymbol{x} \geq 0. \quad (3.2)$$

Now assume that $\alpha_n \to 0$, as $n \to \infty$. The coercivity condition (2.9) implies that

$$\delta' E(\{\boldsymbol{\phi}_n\}) \geq c_1 \left(\varliminf_{n\to\infty} \frac{\beta_n^p}{\alpha_n^2} - c_2 \right).$$

Thus, we need to consider only those strong variations $\{\boldsymbol{\phi}_n\}$ for which

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \qquad \lim_{n\to\infty} \frac{\beta_n^p}{\alpha_n^2} = \gamma < +\infty \quad (3.3)$$

Remark 3.1 *The coercivity condition (2.9) was only needed to reduce the problem of strong local minima to the problem of $W^{1,p}$-local minima. If one is interested only in $W^{1,p}$-local minima, then condition (2.9) is not needed.*

3.2 Decomposition theorem and orthogonality principle

For a strong variation $\{\boldsymbol{\phi}_n\}$ bounded in $W^{1,p}$ we define $\boldsymbol{\zeta}_n = \boldsymbol{\phi}_n/\beta_n$, and $\boldsymbol{\psi}_n = \boldsymbol{\phi}_n/\alpha_n$. We have also a relation $\boldsymbol{\zeta}_n = r_n\boldsymbol{\psi}_n$, where $r_n = \alpha_n/\beta_n \leq 1$.

One of the key tools in our analysis is a version of the Decomposition Theorem due to Kristensen [13], and Fonseca, Müller and Pedregal [7] (see also [9]).

Theorem 3.1 (Decomposition theorem) *Suppose that the sequence of functions* $\boldsymbol{\psi}_n \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ *is bounded in* $W^{1,2}(\Omega;\mathbb{R}^m)$ *and the sequence* $r_n \in (0,1]$ *is such that* $\boldsymbol{\zeta}_n = r_n\boldsymbol{\psi}_n$ *is bounded in* $W^{1,p}(\Omega;\mathbb{R}^m)$, $p \geq 2$*. We also assume that* $r_n = 1$*, if* $p = 2$ *and* $r_n \to 0$*, as* $n \to \infty$*, if* $p > 2$*. Suppose that the sequence* $\alpha_n > 0$ *is such that* $\alpha_n \to 0$*, and* $\alpha_n\boldsymbol{\psi}_n \to 0$*, as* $n \to \infty$*, uniformly in* $\boldsymbol{x} \in \Omega$*. Then there exists a subsequence, not relabeled, sequences of functions* $\boldsymbol{z}_n$ *and* $\boldsymbol{v}_n$ *in* $W_0^{1,\infty}(\Omega;\mathbb{R}^m)$*, and subsets* R_n *of* Ω *such that*

(a) $\boldsymbol{\psi}_n = \boldsymbol{z}_n + \boldsymbol{v}_n$.

(b) For all $\boldsymbol{x} \in \Omega \setminus R_n$ *we have* $\boldsymbol{z}_n(\boldsymbol{x}) = \boldsymbol{\psi}_n(\boldsymbol{x})$ *and* $\nabla\boldsymbol{z}_n(\boldsymbol{x}) = \nabla\boldsymbol{\psi}_n(\boldsymbol{x})$.

(c) The sequence $\{|\boldsymbol{z}_n|^2 + |\nabla\boldsymbol{z}_n|^2\}$ *is equiintegrable.*

(d) $\boldsymbol{v}_n \rightharpoonup 0$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^m)$.

(e) $|R_n| \to 0$*, as* $n \to \infty$.

(f) $\alpha_n\boldsymbol{z}_n \to 0$ *and* $\alpha_n\boldsymbol{v}_n \to 0$ *uniformly in* $\boldsymbol{x} \in \Omega$*, as* $n \to \infty$.

In addition, the sequences $\boldsymbol{t}_n = r_n\boldsymbol{v}_n$ *and* $\boldsymbol{s}_n = r_n\boldsymbol{z}_n$ *satisfy*

(a') $\boldsymbol{\zeta}_n = \boldsymbol{s}_n + \boldsymbol{t}_n$.

(b') For all $\boldsymbol{x} \in \Omega \setminus R_n$ *we have* $\boldsymbol{s}_n(\boldsymbol{x}) = \boldsymbol{\zeta}_n(\boldsymbol{x})$ *and* $\nabla\boldsymbol{s}_n(\boldsymbol{x}) = \nabla\boldsymbol{\zeta}_n(\boldsymbol{x})$.

(c') The sequence $\{|\boldsymbol{s}_n|^p + |\nabla\boldsymbol{s}_n|^p\}$ *is equiintegrable.*

(d') $\boldsymbol{t}_n \rightharpoonup 0$ *weakly in* $W^{1,p}(\Omega;\mathbb{R}^m)$.

We will refer to $\alpha_n\boldsymbol{z}_n$ as the weak part of the variation and to $\alpha_n\boldsymbol{v}_n$ as the strong part. We show that the purely weak part $\{\alpha_n\boldsymbol{z}_n\}$ and purely strong part $\{\alpha_n\boldsymbol{v}_n\}$ of the variation act independently.

Suppose $\boldsymbol{\phi}_n$ is a strong variation such that α_n, β_n, defined by (3.1), satisfy (3.3). Then Theorem 3.1 is applicable to $\boldsymbol{\psi}_n = \boldsymbol{\phi}_n/\alpha_n$, and $r_n = \alpha_n/\beta_n$. Let $\boldsymbol{v}_n$ and $\boldsymbol{z}_n$ be as in Theorem 3.1.

Theorem 3.2 (Orthogonality principle)

$$\mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{v}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{z}_n) \to 0,$$

as $n \to \infty$, strongly in $L^1(\Omega)$.

The orthogonality principle, applied to (1.9), implies that

$$\delta' E(\{\boldsymbol{\phi}_n\}) \geq \varliminf_{n\to\infty} \int_\Omega \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{z}_n) d\boldsymbol{x} + \varliminf_{n\to\infty} \int_\Omega \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x}. \quad (3.4)$$

Thus, in order to prove Theorem 2.2 it will be sufficient to show that each term on the the right-hand side of (3.4) is non-negative.

Proof of Theorem 3.2

Step 1 : Let

$$I_n(\boldsymbol{x}) = \mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{v}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{z}_n).$$

Recall that $\nabla \boldsymbol{v}_n(\boldsymbol{x}) = 0,$ for all $\boldsymbol{x} \in \Omega \backslash R_n$, because $\nabla \boldsymbol{\psi}_n(\boldsymbol{x}) = \nabla \boldsymbol{z}_n,$ for all $\boldsymbol{x} \in \Omega \setminus R_n$. Therefore,

$$\int_\Omega I_n(\boldsymbol{x}) d\boldsymbol{x} = \int_{R_n} I_n(\boldsymbol{x}) d\boldsymbol{x}. \quad (3.5)$$

Then

$$\int_{R_n} |I_n(\boldsymbol{x})| d\boldsymbol{x} \leq \int_{R_n} |\mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{v}_n)| d\boldsymbol{x} + \int_{R_n} |\mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{z}_n)| d\boldsymbol{x} = I_n^{(1)} + I_n^{(2)}. \quad (3.6)$$

Step 2 : We prove $I_n^{(2)} \to 0$, as $n \to \infty$.

Lemma 3.1 *The growth conditions (2.5) and (2.6) together with smoothness of $W(\boldsymbol{F})$ on $\mathcal{R}$ imply that*

$$|\mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G})| \leq C(\mathcal{R}) \Phi(\alpha, \boldsymbol{G}), \quad (3.7)$$

for every $\boldsymbol{F}_0 \in \mathcal{R}$, $\alpha > 0$ and $\boldsymbol{G} \in \mathbb{M}$, where

$$\Phi(\alpha, \boldsymbol{G}) = |\boldsymbol{G}|^2 (1 + |\alpha \boldsymbol{G}|^{p-2}). \quad (3.8)$$

Proof: Observe that

$$\mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G}) = U(\boldsymbol{F}_0, \alpha \boldsymbol{G})|\boldsymbol{G}|^2 + \frac{1}{2}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G}).$$

We apply Lemma 2.1 with $\boldsymbol{H}_1 = \boldsymbol{H}$ and $\boldsymbol{H}_2 = 0$ to get the estimate

$$|U(\boldsymbol{F}_0, \alpha \boldsymbol{G})|\boldsymbol{G}|^2 \leq C(\mathcal{R})\Phi(\alpha, \boldsymbol{G}). \tag{3.9}$$

We also have

$$\frac{1}{2}|(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G})| \leq C(\mathcal{R})|\boldsymbol{G}|^2 \leq C(\mathcal{R})\Phi(\alpha, \boldsymbol{G}).$$

Therefore

$$|\mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G})| \leq C(\mathcal{R})\Phi(\alpha, \boldsymbol{G}).$$

■

Lemma 3.2 *The sequence* $\{\Phi(\alpha_n, \nabla \boldsymbol{z}_n(\boldsymbol{x}))\}$ *is equiintegrable.*

Proof :

From the relation $\beta_n \boldsymbol{s}_n = \alpha_n \boldsymbol{z}_n$, for any $E \subset \Omega$, we have

$$\int_E \Phi(\alpha_n, \nabla \boldsymbol{z}_n(\boldsymbol{x}))d\boldsymbol{x} = \int_E |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 d\boldsymbol{x} + \frac{\beta_n^p}{\alpha_n^2}\int_E |\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p d\boldsymbol{x}. \tag{3.10}$$

The Lemma follows from (3.10), because $|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2$, and $|\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p$ are equi-integrable and the sequence of numbers β_n^p/α_n^2 is bounded.

■

Lemma 3.2 and the inequality (3.7) imply that the second term in the right-hand side of (3.6) converges to 0, because $|R_n| \to 0$.

Step 3 : We now show that $I_n^{(1)}$ converges to 0. By Lemma 2.1 there exists a constant $C(\mathcal{R}) > 0$ such that

$|\mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G}_1) - \mathcal{F}(\boldsymbol{F}_0, \alpha, \boldsymbol{G}_2)| \leq$

$$C(\mathcal{R})(|\boldsymbol{G}_1| + |\boldsymbol{G}_2| + \alpha^{p-2}(|\boldsymbol{G}_1|^{p-1} + |\boldsymbol{G}_2|^{p-1}))|\boldsymbol{G}_1 - \boldsymbol{G}_2| \tag{3.11}$$

for every $\boldsymbol{F}_0 \in \mathcal{R}$, and $\boldsymbol{G}_1, \boldsymbol{G}_2$ in $\mathbb{M}$.

Let

$$d_n(\boldsymbol{x}) = |\mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{\psi}_n) - \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{v}_n)|. \tag{3.12}$$

Then inequality (3.11) implies that

$$d_n(\boldsymbol{x}) \leq C(|\nabla \boldsymbol{\psi}_n| + |\nabla \boldsymbol{v}_n| + \alpha_n^{p-2}(|\nabla \boldsymbol{\psi}_n|^{p-1} + |\nabla \boldsymbol{v}_n|^{p-1}))|\nabla \boldsymbol{z}_n|,$$

for a.e. $\boldsymbol{x} \in \Omega$.

Applying the Cauchy-Schwarz and Hölder inequalities and using the relations

$$\beta_n \boldsymbol{\zeta}_n = \alpha_n \boldsymbol{\psi}_n, \text{ and } \beta_n \boldsymbol{t}_n = \alpha_n \boldsymbol{v}_n,$$

we get

$$\int_{R_n} d_n(\boldsymbol{x}) d\boldsymbol{x} \leq C(\|\nabla \boldsymbol{\psi}_n(\boldsymbol{x})\|_2 + \|\nabla \boldsymbol{v}_n(\boldsymbol{x})\|_2) \left(\int_{R_n} |\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 d\boldsymbol{x} \right)^{1/2} +$$
$$C \frac{\beta_n^p}{\alpha_n^2} (\|\nabla \boldsymbol{\zeta}_n(\boldsymbol{x})\|_p^{p-1} + \|\nabla \boldsymbol{t}_n(\boldsymbol{x})\|_p^{p-1}) \left(\int_{R_n} |\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{1/p}. \tag{3.13}$$

Once again, the equiintegrability of $|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2$ and $|\nabla \boldsymbol{s}_n(\boldsymbol{x})|^p$, and (3.3) imply that $d_n \to 0$, as $n \to \infty$ in $L^1(\Omega)$.

This finishes the proof of the theorem.

■

3.3 Representation formula

Inequality (3.4) reduces our task of proving the non-negativity of $\delta' E(\{\boldsymbol{\phi}_n\})$ to establishing the non-negativity of each individual limit on the the right-hand side of (3.4). In this section we will derive representation formulas for each of the two terms on the the right-hand side of the inequality.

Let us start with the first term

Lemma 3.3 *Assume $\alpha_n \to 0$. Then there exists a subsequence, not relabeled, such that*

$$\lim_{n\to\infty} \int_\Omega \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{z}_n(\boldsymbol{x})) d\boldsymbol{x} = \frac{1}{2} \int_{\overline{\Omega}} \int_{\mathbb{M}} (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})) \boldsymbol{F}, \boldsymbol{F}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) d\boldsymbol{x}, \tag{3.14}$$

where $\{\nu_{\boldsymbol{x}}\}$ *is the Young measure generated by* $\{\nabla \boldsymbol{z}_n(\boldsymbol{x})\}$.

Proof :

We claim that there exists a subsequence, not relabeled, such that

$$\lim_{n\to\infty}\int_{\Omega} U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{z}_n(\boldsymbol{x}))|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 d\boldsymbol{x} = 0, \tag{3.15}$$

and

$$\lim_{n\to\infty}\frac{1}{2}\int_{\Omega}(\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\nabla \boldsymbol{z}_n(\boldsymbol{x}), \nabla \boldsymbol{z}_n(\boldsymbol{x}))d\boldsymbol{x} = \frac{1}{2}\int_{\overline{\Omega}}\int_{\mathbb{M}}(\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\boldsymbol{F}, \boldsymbol{F})d\nu_{\boldsymbol{x}}(\boldsymbol{F})d\boldsymbol{x}. \tag{3.16}$$

Observe that $\{(\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\nabla \boldsymbol{z}_n(\boldsymbol{x}), \nabla \boldsymbol{z}_n(\boldsymbol{x}))\}$ is equiintegrable, since

$$|(\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\nabla \boldsymbol{z}_n(\boldsymbol{x}), \nabla \boldsymbol{z}_n(\boldsymbol{x}))| \le C(\mathcal{R})|\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2.$$

Thus, by the Young measure representation theorem [18, Lemma 6.2] (3.16) holds.

To show (3.9), observe that $\alpha_n \nabla \boldsymbol{z}_n \to 0$ in L^2, because $\nabla \boldsymbol{z}_n$ is bounded in L^2 and $\alpha_n \to 0$. Then we can find a subsequence, not relabeled, such that $\alpha_n \nabla \boldsymbol{z}_n(\boldsymbol{x}) \to 0$, for a.e. $\boldsymbol{x} \in \Omega$. Thus, $U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{z}_n(\boldsymbol{x})) \to U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), 0) = 0$, as $n \to \infty$, for a.e. $\boldsymbol{x} \in \Omega$. Also, using (3.7)

$$|U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{z}_n(\boldsymbol{x}))||\nabla \boldsymbol{z}_n(\boldsymbol{x})|^2 \le C\Phi(\alpha_n, \nabla \boldsymbol{z}_n(\boldsymbol{x})).$$

Lemma 3.2 implies that the sequence $\{U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{z}_n)|\nabla \boldsymbol{z}_n|^2\}$ is equiintegrable. Now, (3.15) follows from the following generalized Vitali convergence theorem [9].

Theorem 3.3 (Generalized Vitali convergence theorem) *Let* $(X, \mathfrak{M}, \mu)$ *be a positive measure space. If (i)* $\mu(X)$ *is finite, (ii)* $f_n \to 0$*, a.e. as* $n \to \infty$*, (iii)* g_n *is bounded in* $L^1(\mu)$ *and, (iv) the sequence* $\{f_n g_n\}$ *is equiintegrable. Then* $f_n g_n \to 0$ *in* $L^1(\mu)$.

This finishes the proof of Lemma 3.3. ■

The next step is to characterize up to a subsequence

$$\lim_{n\to\infty}\int_\Omega \mathcal{F}(\nabla \boldsymbol{y}_0, \alpha_n, \nabla \boldsymbol{v}_n)d\boldsymbol{x} = \lim_{n\to\infty}\frac{1}{2}\int_\Omega (\mathsf{L}(\nabla \boldsymbol{y}_0)\nabla \boldsymbol{v}_n, \nabla \boldsymbol{v}_n)d\boldsymbol{x} + \lim_{n\to\infty}\int_\Omega U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{v}_n)|\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x}. \tag{3.17}$$

The first term in the right-hand side of (3.17) can not be written in terms of Young measures because the sequence $\{\nabla \boldsymbol{v}_n\}$ is not equiintegrable. Instead, consider the $\mathbb{R}^{m\times d}$ valued measures on Ω given by

$$\boldsymbol{\mu}_n = \nabla \boldsymbol{v}_n(\boldsymbol{x})|\nabla \boldsymbol{v}_n(\boldsymbol{x})|$$

with polar decomposition

$$d\boldsymbol{\mu}_n = \frac{\nabla \boldsymbol{v}_n(\boldsymbol{x})}{|\nabla \boldsymbol{v}_n(\boldsymbol{x})|}d\pi_n(\boldsymbol{x}),$$

where $d\pi_n(\boldsymbol{x}) = |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^2 d\boldsymbol{x}$.

Then we can define a sequence of measures on a separable space $C(\overline{\Omega} \times \mathcal{R} \times S)$, where S is a unit sphere in $\mathbb{M}$ space, by

$$\Lambda_n(f) = \int_\Omega f(\boldsymbol{x}, \nabla \boldsymbol{y}_0(\boldsymbol{x}), \frac{\nabla \boldsymbol{v}_n(\boldsymbol{x})}{|\nabla \boldsymbol{v}_n(\boldsymbol{x})|})d\pi_n(\boldsymbol{x}). \tag{3.18}$$

Observe that

$$|\Lambda_n(f)| \le \|f\|_{C(\overline{\Omega}\times\mathcal{R}\times S)}\|\nabla \boldsymbol{v}_n\|_2^2.$$

Therefore, Λ_n is a bounded sequence of linear continuous functionals on $C(\overline{\Omega}\times \mathcal{R}\times S)$, since $\{\nabla \boldsymbol{v}_n\}$ is bounded in $L^2(\Omega;\mathbb{M})$. By the Banach-Alaoglu theorem there exists a subsequence, not relabeled, $\{\Lambda_n\}$ and a linear continuous functional Λ on $C(\overline{\Omega}\times\mathcal{R}\times S)$ such that $\Lambda_n \rightharpoonup \Lambda$ weak$-*$. By Riesz representation theorem there exists a non-negative Radon measure M on $\overline{\Omega} \times \mathcal{R} \times S$ such that for every $f \in C(\overline{\Omega} \times \mathcal{R} \times S)$

$$\lim_{n\to\infty}\int_\Omega f(\boldsymbol{x}, \nabla \boldsymbol{y}_0(\boldsymbol{x}), \frac{\nabla \boldsymbol{v}_n(\boldsymbol{x})}{|\nabla \boldsymbol{v}_n(\boldsymbol{x})|})d\pi_n(\boldsymbol{x}) = \int_{\overline{\Omega}\times\mathcal{R}\times S} f(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G})dM(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}).$$

Let π be the projection of M on to $\overline{\Omega}$. Then by [5, Proposition 3.1] there exists a family of probability measures $\{\lambda_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Omega}$ on $S \times \mathcal{R}$ such that for every $f \in C(\overline{\Omega} \times \mathcal{R} \times S)$

$$\int_{\overline{\Omega}\times\mathcal{R}\times S} f(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}) dM(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}) = \int_{\overline{\Omega}} \left[\int_{\mathcal{R}\times S} f(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}) d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0, \boldsymbol{G}) \right] d\pi(\boldsymbol{x}). \quad (3.19)$$

For $f(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}) = (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G})$, we have

$$\lim_{n\to\infty} \int_\Omega (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})) \frac{\nabla \boldsymbol{v}_n(\boldsymbol{x})}{|\nabla \boldsymbol{v}_n(\boldsymbol{x})|}, \frac{\nabla \boldsymbol{v}_n(\boldsymbol{x})}{|\nabla \boldsymbol{v}_n(\boldsymbol{x})|}) |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\overline{\Omega}} \left[\int_{\mathcal{R}\times S} (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G}) d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0, \boldsymbol{G}) \right] d\pi(\boldsymbol{x}). \quad (3.20)$$

Therefore,

$$\lim_{n\to\infty} \int_\Omega (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})) \nabla \boldsymbol{v}_n(\boldsymbol{x}), \nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x} = \int_{\overline{\Omega}} \left[\int_{\mathcal{R}\times S} (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G}) d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0, \boldsymbol{G}) \right] d\pi(\boldsymbol{x}). \quad (3.21)$$

Remark 3.2 *If $f(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{G}) = \xi(\boldsymbol{x}) \in C(\overline{\Omega})$ depends only on $\boldsymbol{x}$, then*

$$\underline{\lim}_{n\to\infty} \int_\Omega \boldsymbol{\xi}(\boldsymbol{x}) |\nabla \boldsymbol{v}_n(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\overline{\Omega}} \boldsymbol{\xi}(\boldsymbol{x}) d\pi(\boldsymbol{x}).$$

Which implies that $d\pi_n \rightharpoonup d\pi$ in the sense of measures.

In order to compute the second limit in the right-hand side of (3.17) we rewrite the integrand in terms of the bounded and continuous function $B(\boldsymbol{F}_0, \boldsymbol{H})$ given by

$$B(\boldsymbol{F}_0, \boldsymbol{H}) = \frac{U(\boldsymbol{F}_0, \boldsymbol{H})}{1 + |\boldsymbol{H}|^{p-2}}. \quad (3.22)$$

Thus, we have

$$\lim_{n\to\infty} \int_\Omega U(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{v}_n) |\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x} = \lim_{n\to\infty} \int_\Omega B(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n \nabla \boldsymbol{v}_n(\boldsymbol{x})) \Phi(\alpha_n, \nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x}. \quad (3.23)$$

Following DiPerna and Majda [3, Theorem 4.1] we prove the following lemma.

Lemma 3.4 *Let* $C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$ *denote the set of all continuous and bounded functions on* $\overline{\Omega}\times\mathcal{R}\times\mathbb{M}$. *There exist a subsequence, not relabeled, a nonnegative measure* σ *on* $\overline{\Omega}$ *and a continuous linear transformation* $\mathsf{T}: C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M}) \to L^\infty_\sigma(\overline{\Omega})$ *such that for any* $B \in C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$

$$\lim_{n\to\infty}\int_\Omega B(\boldsymbol{x}, \nabla\boldsymbol{y}_0(\boldsymbol{x}), \alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x}))\Phi(\alpha_n, \nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x} = \int_{\overline{\Omega}}(\mathsf{T}B)(\boldsymbol{x})d\sigma(\boldsymbol{x}). \tag{3.24}$$

Proof :

For each fixed n, the functional

$$\Lambda_n(B) = \int_\Omega B(\boldsymbol{x}, \nabla\boldsymbol{y}_0(\boldsymbol{x}), \alpha_n\nabla\boldsymbol{v}_n(\boldsymbol{x}))\Phi(\alpha_n, \nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}$$

is a linear and continuous functional on $C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$. And

$$|\Lambda_n(B)| \le \|B\|_\infty(\|\nabla\boldsymbol{v}_n\|_2^2 + \alpha_n^{p-2}\|\nabla\boldsymbol{v}_n\|_p^p) = \|B\|_\infty(\|\nabla\boldsymbol{v}_n\|_2^2 + \frac{\beta_n^p}{\alpha_n^2}\|\nabla\boldsymbol{t}_n\|_p^p)$$

implies that Λ_n is a bounded sequence. By the Banach-Alaoglu theorem there exists a subsequence, not relabeled, and a linear continuous functional Λ on $C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$ such that $\Lambda_n \rightharpoonup \Lambda$ weak$-*$.

Observe that $a \mapsto \Lambda(a)$ is a linear continuous functional on $C(\overline{\Omega})$. Therefore there exists a Radon measure σ on $\overline{\Omega}$ such that

$$\Lambda(a) = \int_{\overline{\Omega}} a(\boldsymbol{x})d\sigma(\boldsymbol{x}). \tag{3.25}$$

Therefore, σ is a weak$-*$ limit of $\Phi(\alpha_n, \nabla\boldsymbol{v}_n)d\boldsymbol{x}$ in the sense of measures.

Let us fix $B \in C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$ and observe that the functional $a \mapsto \Lambda(aB)$ is a continuous linear functional on $C(\overline{\Omega})$. Hence, there exists a Radon measure M_B such that

$$\Lambda(aB) = \int_{\overline{\Omega}} a(\boldsymbol{x})dM_B(\boldsymbol{x}),$$

and we have

$$|\Lambda(aB)| \le \|B\|_\infty\int_{\overline{\Omega}}|a(\boldsymbol{x})|d\sigma(\boldsymbol{x}). \tag{3.26}$$

Lemma 3.5 *For each* $B \in C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M})$, *the measure* M_B *is absolutely continuous with respect to* σ.

Proof :

Let $B \in C_B(\overline{\Omega} \times \mathcal{R} \times \mathbb{M})$, $B \geq 0$ be fixed and $a \in C(\overline{\Omega})$, $a \geq 0$.

$$\Lambda(aB) \leq \lim_{n\to\infty} \int_{\overline{\Omega}} a(\boldsymbol{x})\|B\|_\infty \Phi(\alpha_n, \nabla \boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x} = \|B\|_\infty \int_{\overline{\Omega}} a(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

That is

$$\int_{\overline{\Omega}} a(\boldsymbol{x})dM_B(\boldsymbol{x}) \leq \|B\|_\infty \int_{\overline{\Omega}} a(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

Which implies that

$$\int_{\overline{\Omega}} a(\boldsymbol{x})d(\|B\|_\infty\sigma - M_B)(\boldsymbol{x}) \geq 0.$$

Therefore the measure $\|B\|_\infty\sigma - M_B$ is non-negative. Thus, for all Borel subsets $E \subset \overline{\Omega}$ $\|B\|_\infty\sigma(E) - M_B(E) \geq 0$.

If $\sigma(E) = 0$, then $M_B(E) \leq 0$. This implies that $M_B(E) = 0$, because M_B is a non-negative measure for $B \geq 0$. If B is not positive, then we can write $B = B^+ - B^-$, where $B^\pm$ are both bounded and non-negative. M_B is then absolutely continuous with respect to σ, because $M_{B^\pm}$ are, and $M_B = M_{B^+} - M_{B^-}$. This finishes the proof of Lemma 3.5.

■

By Lemma 3.5 and the Radon-Nikodym theorem there exists a function $f_B \in L^1_\sigma(\overline{\Omega})$ such that for every Borel subset E of $\overline{\Omega}$

$$M_B(E) = \int_E f_B(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

The integrand f_B linearly depends on B and inequality (3.26), together with density of $C(\overline{\Omega})$ in $L^1_\sigma(\overline{\Omega})$ implies that $\|f_B\|_{L^\infty_\sigma(\overline{\Omega})} \leq \|B\|_\infty$. Hence, the map $B \mapsto f_B$ defines a bounded linear transformation $\mathsf{T} : C_B(\overline{\Omega}\times\mathcal{R}\times\mathbb{M}) \to L^\infty_\sigma(\overline{\Omega})$. Therefore,

$$\Lambda(B) = \int_{\overline{\Omega}} (\mathsf{T}B)(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

■

Lemma 3.6 *The operator* T *has the following properties:*

(a) $|\mathsf{T}B| \leq \mathsf{T}|B|$, *for any* $B \in C_B(\overline{\Omega} \times \mathcal{R} \times \mathbb{M})$;

(b) $(\mathsf{T}a)(\boldsymbol{x}) = a(\boldsymbol{x})$, *for any* $a \in C(\overline{\Omega})$.

Proof : Property (a) follows from $|B| - B \geq 0$ and the non-negativity of $\Phi(\alpha, \boldsymbol{G})$. To prove property (b), take $B(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{F}) = a(\boldsymbol{x})$ in Lemma 3.4. Using (3.25), we obtain

$$\int_{\Omega} a(\boldsymbol{x}) d\sigma(\boldsymbol{x}) = \Lambda(a) = \lim_{n\to\infty} \int_{\Omega} a(\boldsymbol{x}) \Phi(\alpha_n, \nabla \boldsymbol{v}_n(\boldsymbol{x})) d\boldsymbol{x} = \int_{\Omega} (\mathsf{T}a)(\boldsymbol{x}) d\sigma(\boldsymbol{x}),$$

for any test function $a \in C(\overline{\Omega})$. Therefore, $(\mathsf{T}a)(\boldsymbol{x}) = a(\boldsymbol{x})$.

■

Applying Lemma 3.4 to

$$b_0(\boldsymbol{F}) = \frac{1}{1 + |\boldsymbol{F}|^{p-2}},$$

we obtain

$$d\pi_n = |\nabla \boldsymbol{v}_n|^2 d\boldsymbol{x} = b_0(\alpha_n \nabla \boldsymbol{v}_n) \Phi(\alpha_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x} \rightharpoonup (\mathsf{T}b_0)(\boldsymbol{x}) d\sigma,$$

where the convergence is in the sense of weak-$*$ topology on $C(\overline{\Omega})^*$. Thus, π is an absolutely continuous measure with respect to σ and

$$d\pi = (\mathsf{T}b_0)(\boldsymbol{x}) d\sigma. \tag{3.27}$$

Combining (3.21), (3.24), and (3.27) we have the following representation formula.

$$\lim_{n\to\infty} \int_{\Omega} \mathcal{F}(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \alpha_n, \nabla \boldsymbol{v}_n) d\boldsymbol{x} = \int_{\Omega} \mathcal{I}(\boldsymbol{x}) d\sigma(\boldsymbol{x}), \tag{3.28}$$

where

$$\mathcal{I}(\boldsymbol{x}) = (\mathsf{T}B)(\boldsymbol{x}) + \frac{(\mathsf{T}b_0)(\boldsymbol{x})}{2} \left(\int_{\mathcal{R}\times\mathcal{S}} (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G}) d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0, \boldsymbol{G}) \right). \tag{3.29}$$

■

In order to finish the proof of the sufficiency theorem it remains to prove that (3.14) and (3.28) are non-negative. We will show that the non-negativity of (3.14) follows from the non-negativity of the second variation and the non-negativity of (3.28) follows from condition (2.13). However, (3.28) has a local character, but condition(2.13) has global character. In order to reduce one to the other, we need the following localization principle.

3.4 Localization principle

Theorem 3.4 *Let* $\boldsymbol{a} \in \overline{\Omega}$. *Let* $\theta_k \in C_0^\infty(B(0,1))$ *be the cut-off functions constructed in such a way that* $\theta_k(\boldsymbol{x}) = 1$, *if* $|\boldsymbol{x}| < 1 - 1/k$, $\theta_k(\boldsymbol{x}) = 0$, *if* $|\boldsymbol{x}| \geq 1$, *and* $\theta_k(\boldsymbol{x}) \in [0,1]$, *for all* $\boldsymbol{x} \in \mathbb{R}^d$ *and such that* $\|\nabla\theta_k\|_\infty \leq Ck$, *and* $\theta_k(\boldsymbol{x}) \to \chi_{B(0,1)}(\boldsymbol{x})$, *for all* $\boldsymbol{x} \in \mathbb{R}^d$.

Then

$$\mathcal{I}(\boldsymbol{a}) = \lim_{r\to 0}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\sigma(B_\Omega(\boldsymbol{a},r))}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\nabla(\theta_{k,r}(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})))d\boldsymbol{x} \tag{3.30}$$

for σ*-almost every* $\boldsymbol{a} \in \overline{\Omega}$, *where* $\mathcal{I}(\boldsymbol{a})$ *is given by (3.29),*

$$B_\Omega(\boldsymbol{a},r) = \{\boldsymbol{x} \in \overline{\Omega} : |\boldsymbol{x}-\boldsymbol{a}| < r\}$$

and

$$\theta_{k,r}(\boldsymbol{x}) = \theta_k\left(\frac{\boldsymbol{x}-\boldsymbol{a}}{r}\right), \boldsymbol{x} \in B_\Omega(\boldsymbol{a},r).$$

Observe that $\theta_{k,r} \in C_0^\infty(B_\Omega(\boldsymbol{a},r))$ with $\|\nabla\theta_{k,r}\|_\infty \leq Ck/r$, and

$$\lim_{k\to\infty}\theta_{k,r}(\boldsymbol{z}) = \chi_{B_\Omega(\boldsymbol{a},r)}(\boldsymbol{z}).$$

Proof :

Lemma 3.7 *For each fixed* k *and* r

$$\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\nabla(\theta_{k,r}(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x})))d\boldsymbol{x} = \lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}.$$

Proof : Let

$$T_{n,k,r}(\boldsymbol{x}) = \mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\nabla(\theta_{k,r}\boldsymbol{v}_n(\boldsymbol{x}))) - \mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x})).$$

We show that

$$\int_{B_\Omega(\boldsymbol{a},r)}|T_{n,k,r}(\boldsymbol{x})|d\boldsymbol{x} \to 0, \text{ as } n\to\infty.$$

By using inequality (3.11)

$$|T_{n,k,r}(\boldsymbol{x})| \leq C(|\nabla(\theta_{k,r}\boldsymbol{v}_n)| + |\theta_{k,r}\boldsymbol{v}_n|) + \\ C(\alpha_n^{p-2}(|\nabla(\theta_{k,r}\boldsymbol{v}_n)|^{p-1} + |\theta_{k,r}\boldsymbol{v}_n|^{p-1}))|\nabla(\theta_{k,r}\boldsymbol{v}_n) - \theta_{k,r}\nabla\boldsymbol{v}_n|, \quad (3.31)$$

for some constant $C > 0$. Observe that

$$\nabla(\theta_{k,r}\boldsymbol{v}_n(\boldsymbol{x})) = \boldsymbol{v}_n(\boldsymbol{x}) \otimes \nabla\theta_{k,r}(\boldsymbol{x}) + \theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}).$$

Substituting the above identity in (3.31), we get

$$|T_{n,k,r}(\boldsymbol{x})| \leq C(|\boldsymbol{v}_n \otimes \nabla\theta_{k,r} + \theta_{k,r}\nabla\boldsymbol{v}_n| + |\theta_{k,r}\boldsymbol{v}_n| + \\ C(\alpha_n^{p-2}(|\boldsymbol{v}_n \otimes \nabla\theta_{k,r} + \theta_{k,r}\nabla\boldsymbol{v}_n|^{p-1} + |\theta_{k,r}\boldsymbol{v}_n|^{p-1})|\boldsymbol{v}_n \otimes \nabla\theta_{k,r}|). \quad (3.32)$$

From the relation $\beta_n\boldsymbol{t}_n = \alpha_n\boldsymbol{v}_n$ we have

$$|T_{n,k,r}(\boldsymbol{x})| \leq C'(k,r)(2|\boldsymbol{v}_n|^2 + 2|\nabla\boldsymbol{v}_n||\boldsymbol{v}_n|) + \\ C'(k,r)(\frac{\beta_n^p}{\alpha_n^2}\{|\boldsymbol{t}_n \otimes \nabla\theta_{k,r} + \theta_{k,r}\nabla\boldsymbol{t}_n|^{p-1} + |\boldsymbol{t}_n \otimes \nabla\theta_{k,r}|^{p-1}\}|\boldsymbol{t}_n \otimes \nabla\theta_{k,r}|).$$

Therefore, since $\nabla\theta_{k,r}$ and $\theta_{k,r}$ are bounded for fixed r and k,

$$\int_\Omega |T_{n,k,r}(\boldsymbol{x})|d\boldsymbol{x} \leq C(k,r)\left(\|\boldsymbol{v}_n\|_2 + \|\nabla\boldsymbol{v}_n\|_2\|\boldsymbol{v}_n\|_2\right) + \\ C(k,r)\frac{\beta_n^p}{\alpha_n^2}\left\{[\|\boldsymbol{t}_n\|_p + \|\nabla\boldsymbol{t}_n\|_p]^{p-1} + \|\nabla\boldsymbol{t}_n\|_p^{p-1}\right\}\|\boldsymbol{t}_n\|_p. \quad (3.33)$$

The right-hand side of (3.33) converges to 0, because $\boldsymbol{v}_n \rightharpoonup 0$ in $W^{1,2}$, $\boldsymbol{t}_n \rightharpoonup 0$ in $W^{1,p}$ and $\frac{\beta_n^p}{\alpha_n^2}$ is a bounded sequence.

■

Therefore to prove Theorem 3.4 it is enough to show that

$$\mathcal{I}(\boldsymbol{a}) = \lim_{r\to0}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\sigma(B_\Omega(\boldsymbol{a},r))}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}, \quad (3.34)$$

for σ-almost every $\boldsymbol{a} \in \overline{\Omega}$.

Lemma 3.8 *For each fixed k and r*

$$\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}=$$
$$\int_{B_\Omega(\boldsymbol{a},r)}\left[(\mathsf{T}B_{k,r})(\boldsymbol{x})+\frac{\theta_{k,r}^2(\boldsymbol{x})(\mathsf{T}b_0)(\boldsymbol{x})}{2}\int_{\mathcal{R}\times S}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G},\boldsymbol{G})d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0,\boldsymbol{G})\right]d\sigma(\boldsymbol{x}),\tag{3.35}$$

where

$$B_{k,r}(\boldsymbol{x},\boldsymbol{F}_0,\boldsymbol{H})=\frac{\theta_{k,r}^2(\boldsymbol{x})(1+|\theta_{k,r}(\boldsymbol{x})\boldsymbol{H}|^{p-2})}{1+|\boldsymbol{H}|^{p-2}}B(\boldsymbol{F}_0,\theta_{k,r}(\boldsymbol{x})\boldsymbol{H}).\tag{3.36}$$

Proof : Observe that

$$\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))=\theta_{k,r}^2(\boldsymbol{x})U(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n\theta_{k,r}\nabla\boldsymbol{v}_n)|\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2+$$
$$\frac{\theta_{k,r}^2(\boldsymbol{x})}{2}\left(\mathsf{L}(\nabla\boldsymbol{y}_0(\boldsymbol{x}))\nabla\boldsymbol{v}_n(\boldsymbol{x}),\nabla\boldsymbol{v}_n(\boldsymbol{x})\right).$$

And

$$\theta_{k,r}^2(\boldsymbol{x})U(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n\theta_{k,r}\nabla\boldsymbol{v}_n)|\nabla\boldsymbol{v}_n(\boldsymbol{x})|^2$$
$$=\frac{\theta_{k,r}^2(\boldsymbol{x})(1+|\alpha_n\theta_{k,r}\nabla\boldsymbol{v}_n|^{p-2})}{1+|\alpha_n\nabla\boldsymbol{v}_n|^{p-2}}B(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n\theta_{k,r}\nabla\boldsymbol{v}_n)\Phi(\alpha_n,\nabla\boldsymbol{v}_n)$$

Therefore

$$\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))=$$
$$B_{k,r}(\boldsymbol{x},\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n\nabla\boldsymbol{v}_n)\Phi(\alpha_n,\nabla\boldsymbol{v}_n)+\frac{\theta_{k,r}^2(\boldsymbol{x})}{2}\left(\mathsf{L}(\nabla\boldsymbol{y}_0(\boldsymbol{x}))\nabla\boldsymbol{v}_n,\nabla\boldsymbol{v}_n\right).\tag{3.37}$$

Applying the representation formula (3.28) we obtain

$$\lim_{n\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{F}(\nabla\boldsymbol{y}_0(\boldsymbol{x}),\alpha_n,\theta_{k,r}(\boldsymbol{x})\nabla\boldsymbol{v}_n(\boldsymbol{x}))d\boldsymbol{x}=$$
$$\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{I}_{k,r}(\boldsymbol{x})d\sigma(\boldsymbol{x}),\tag{3.38}$$

where

$$\mathcal{I}_{k,r}(\boldsymbol{x})=(\mathsf{T}B_{k,r})(\boldsymbol{x})+$$
$$\frac{\theta_{k,r}^2(\boldsymbol{x})(\mathsf{T}b_0)(\boldsymbol{x})}{2}\int_{\mathcal{R}\times S}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G},\boldsymbol{G})d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0,\boldsymbol{G})\tag{3.39}$$

■

Next we show that

Lemma 3.9

$$\lim_{k\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{I}_{k,r}(\boldsymbol{x})d\sigma(\boldsymbol{x})=\int_{B_\Omega(\boldsymbol{a},r)}\mathcal{I}(\boldsymbol{x})d\sigma(\boldsymbol{x}),\tag{3.40}$$

where

$$\mathcal{I}(\boldsymbol{x})=(\mathsf{T}B)(\boldsymbol{x})+\frac{(\mathsf{T}b_0)(\boldsymbol{x})}{2}\int_{\mathcal{R}\times S}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G},\boldsymbol{G})d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0,\boldsymbol{G}).\tag{3.41}$$

Proof : For every $\boldsymbol{x}\in B_\Omega(\boldsymbol{a},r)$, we have

$$\lim_{k\to\infty}\left(\frac{\theta_{k,r}^2(\boldsymbol{x})(\mathsf{T}b_0)(\boldsymbol{x})}{2}\int_{\mathcal{R}\times S}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G},\boldsymbol{G})d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0,\boldsymbol{G})\right)=\frac{(\mathsf{T}b_0)(\boldsymbol{x})}{2}\int_{\mathcal{R}\times S}(\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G},\boldsymbol{G})d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0,\boldsymbol{G}).\tag{3.42}$$

Let us show that

$$\lim_{k\to\infty}\int_{B_\Omega(\boldsymbol{a},r)}(\mathsf{T}B_{k,r})(\boldsymbol{x})d\sigma(\boldsymbol{x})=\int_{B_\Omega(\boldsymbol{a},r)}(\mathsf{T}B)(\boldsymbol{x})d\sigma(\boldsymbol{x}).\tag{3.43}$$

Lemma 3.10 *For each* $\boldsymbol{x}\in B_\Omega(\boldsymbol{a},r)$,

$$\lim_{k\to\infty}B_{k,r}(\boldsymbol{x},\boldsymbol{F}_0,\boldsymbol{H})=B(\boldsymbol{F}_0,\boldsymbol{H}),$$

uniformly in $(\boldsymbol{F}_0,\boldsymbol{H})\in\mathcal{R}\times\mathbb{M}$.

Proof :

Claim 1 : Let $q>0$, $\theta_k\geq 0$ and $\theta_k\to\theta_0$, as $k\to\infty$. Then $\theta_k^qB(\boldsymbol{F}_0,\theta_k\boldsymbol{H})\to\theta_0^qB(\boldsymbol{F}_0,\theta_0\boldsymbol{H})$, uniformly in $(\boldsymbol{F}_0,\boldsymbol{H})\in\mathcal{R}\times\mathbb{M}$.

If $\theta_0=0$, claim is true, because $B(\boldsymbol{F}_0,\boldsymbol{H})$ is bounded. Assume $\theta_0>0$. Suppose to the contrary

$$\varlimsup_{k\to\infty}\sup_{(\boldsymbol{F}_0,\boldsymbol{H})\in\mathcal{R}\times\mathbb{M}}|\theta_k^qB(\boldsymbol{F}_0,\theta_k\boldsymbol{H})-\theta_0^qB(\boldsymbol{F}_0,\theta_0\boldsymbol{H})|>0.\tag{3.44}$$

Then there exists a sequence $(\boldsymbol{F}_0^{(k)},\boldsymbol{H}_k)\in\mathcal{R}\times\mathbb{M}$ such that $|\boldsymbol{H}_k|\to\infty$ and

$$\varlimsup_{k\to\infty}|\theta_k^qB(\boldsymbol{F}_0^{(k)},\theta_k\boldsymbol{H}_k)-\theta_0^qB(\boldsymbol{F}_0^{(k)},\theta_0\boldsymbol{H}_k)|>0.$$

Claim 2 :

$$B(\boldsymbol{F}_0^{(k)}, \theta_k \boldsymbol{H}_k) - B(\boldsymbol{F}_0^{(k)}, \theta_0 \boldsymbol{H}_k) \to 0, \text{ as } k \to \infty.$$

Using condition $\boldsymbol{C2}$, we obtain

$$|B_{\boldsymbol{H}}(\boldsymbol{F}_0, \boldsymbol{H})| \leq \frac{C(\mathcal{R})}{1 + |\boldsymbol{H}|},$$

for some constant $C(\mathcal{R}) > 0$. Applying Lagrange's mean value theorem

$$|B(\boldsymbol{F}_0^{(k)}, \theta_k \boldsymbol{H}_k) - B(\boldsymbol{F}_0^{(k)}, \theta_0 \boldsymbol{H}_k)| = |(B_{\boldsymbol{H}}(\boldsymbol{F}_0^{(k)}, \boldsymbol{\xi}), \boldsymbol{H}_k)(\theta_k - \theta_0)|, \qquad (3.45)$$

where $\boldsymbol{\xi} = t_k \theta_k \boldsymbol{H}_k + (1 - t_k)\theta_0 \boldsymbol{H}_k = \theta_0 \boldsymbol{H}_k + (\theta_k - \theta_0) t_k \boldsymbol{H}_k$, for some $t_k \in [0, 1]$.

Since $|\boldsymbol{\xi}| \geq \theta_0 |\boldsymbol{H}_k| - |\theta_k - \theta_0| \boldsymbol{H}_k|$,

$$|B(\boldsymbol{F}_0^{(k)}, \theta_k \boldsymbol{H}_k) - B(\boldsymbol{F}_0^{(k)}, \theta_0 \boldsymbol{H}_k)| \leq \frac{C(\mathcal{R})}{1 + |\boldsymbol{\xi}|} |\boldsymbol{H}_k||\theta_k - \theta_0| \leq$$

$$\frac{C(\mathcal{R})|\theta_k - \theta_0|}{1 + \theta_0 - |\theta_k - \theta_0|} \to 0, \text{ as } k \to \infty.$$

This finishes the proof of Claim 2.

Now

$$|\theta_k^q B(\boldsymbol{F}_0^{(k)}, \theta_k \boldsymbol{H}_k) - \theta_0^q B(\boldsymbol{F}_0^{(k)}, \theta_0 \boldsymbol{H}_k)| \leq$$
$$|\theta_k^q - \theta_0^q| \|B\|_\infty + \theta_0^q |B(\boldsymbol{F}_0^{(k)}, \theta_k \boldsymbol{H}_k) - B(\boldsymbol{F}_0^{(k)}, \theta_0 \boldsymbol{H}_k)| \qquad (3.46)$$

Taking a limit as $k \to \infty$ in (3.46) we get a contradition to (3.44).

■

Lemma 3.9 follows from the bounded convergence theorem and the following lemma.

Lemma 3.11

$$\lim_{k \to \infty} (\mathsf{T} B_{k,r})(\boldsymbol{x}) = (\mathsf{T} B)(\boldsymbol{x}),$$

for $\sigma-$ *a.e.* $\boldsymbol{x} \in B_\Omega(\boldsymbol{a}, r)$.

Proof :

We have $B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H})$ are uniformly bounded.

Let

$$\delta_k(\boldsymbol{x}) = \sup_{(\boldsymbol{F}_0,\boldsymbol{H})\in\mathcal{R}\times\mathbb{M}} |B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H}) - B(\boldsymbol{F}_0, \boldsymbol{H})|.$$

$\delta_k(\boldsymbol{x})$ are uniformly bounded functions such that $\delta_k(\boldsymbol{x}) \to 0$. The continuity of $\delta_k(\boldsymbol{x})$ follows from the uniform continuity of $B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H})$ in the sense of definition 4 below.

Definition 4 *We say $B \in C_B(\overline{\Omega} \times \mathcal{R} \times \mathbb{M})$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(\boldsymbol{F}_0, \boldsymbol{H}) \in \mathcal{R}\times\mathbb{M}$ and $\boldsymbol{x}', \boldsymbol{x}''$, such that $|\boldsymbol{x}' - \boldsymbol{x}''| < \delta$,*

$$|B(\boldsymbol{x}', \boldsymbol{F}_0, \boldsymbol{H}) - B(\boldsymbol{x}'', \boldsymbol{F}_0, \boldsymbol{H})| < \epsilon.$$

Lemma 3.12 *$B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H})$ are uniformly continuous in the sense of definition 4.*

Proof: Recall that

$$B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H}) = \frac{\theta^2_{k,r}(\boldsymbol{x})(1 + |\theta_{k,r}(\boldsymbol{x})\boldsymbol{H}|^{p-2})}{1 + |\boldsymbol{H}|^{p-2}} B(\boldsymbol{F}_0, \theta_{k,r}(\boldsymbol{x})\boldsymbol{H}) =$$
$$\frac{\theta^2_{k,r}(\boldsymbol{x})}{1 + |\boldsymbol{H}|^{p-2}} B(\boldsymbol{F}_0, \theta_{k,r}(\boldsymbol{x})\boldsymbol{H}) + \frac{\theta^p_{k,r}(\boldsymbol{x})|\boldsymbol{H}|^{p-2}}{1 + |\boldsymbol{H}|^{p-2}} B(\boldsymbol{F}_0, \theta_{k,r}(\boldsymbol{x})\boldsymbol{H}) \quad (3.47)$$

Suppose $\{\boldsymbol{x}'_n, \boldsymbol{x}''_n\} \subset \overline{\Omega}$, such that

$$\lim_{n\to\infty} \boldsymbol{x}'_n = \lim_{n\to\infty} \boldsymbol{x}''_n = \boldsymbol{x}^* \in \overline{\Omega}.$$

Let $\theta'_n = \theta_{k,r}(\boldsymbol{x}'_n)$, $\theta''_n = \theta_{k,r}(\boldsymbol{x}''_n)$, and $\theta_0 = \theta_{k,r}(\boldsymbol{x}^*)$. Then

$$|B_{k,r}(\boldsymbol{x}'_n, \boldsymbol{F}_0, \boldsymbol{H}) - B_{k,r}(\boldsymbol{x}''_n, \boldsymbol{F}_0, \boldsymbol{H})| \le |(\theta'_n)^2 B(\boldsymbol{F}_0, \theta'_n\boldsymbol{H}) - (\theta''_n)^2 B(\boldsymbol{F}_0, \theta''_n\boldsymbol{H})| +$$
$$|(\theta'_n)^p B(\boldsymbol{F}_0, \theta'_n\boldsymbol{H}) - (\theta''_n)^p B(\boldsymbol{F}_0, \theta''_n\boldsymbol{H})| \le$$
$$|(\theta'_n)^2 B(\boldsymbol{F}_0, \theta'_n\boldsymbol{H}) - (\theta_0)^2 B(\boldsymbol{F}_0, \theta_0\boldsymbol{H})| + |(\theta_0)^2 B(\boldsymbol{F}_0, \theta_0\boldsymbol{H}) - (\theta''_n)^2 B(\boldsymbol{F}_0, \theta''_n\boldsymbol{H})| +$$
$$|(\theta'_n)^p B(\boldsymbol{F}_0, \theta'_n\boldsymbol{H}) - (\theta_0)^p B(\boldsymbol{F}_0, \theta_0\boldsymbol{H})| + |(\theta_0)^p B(\boldsymbol{F}_0, \theta_0\boldsymbol{H}) - (\theta''_n)^p B(\boldsymbol{F}_0, \theta''_n\boldsymbol{H})|. \quad (3.48)$$

Applying Lemma 3.10, we obtain uniform continuity of $B_{k,r}(\boldsymbol{x}, \boldsymbol{F}_0, \boldsymbol{H})$, and hence continuity of $\delta_k(\boldsymbol{x})$.

■

Now, applying the properties of operator T from Lemma 3.6 we have

$$|(\mathsf{T}B_{k,r})(\boldsymbol{x}) - (\mathsf{T}B)(\boldsymbol{x})| \leq \mathsf{T}|B_{k,r}(\boldsymbol{x}) - B(\boldsymbol{x})| \leq \mathsf{T}\delta_k(\boldsymbol{x}) = \delta_k(\boldsymbol{x}),$$

for $\sigma-$ a.e.$\boldsymbol{x} \in B_\Omega(\boldsymbol{a}, r)$. Observe that

$$\left(\frac{\theta_{k,r}^2(\boldsymbol{x})(\mathsf{T}b_0)(\boldsymbol{x})}{2} \int_{\mathcal{R}\times S} (\mathsf{L}(\boldsymbol{F}_0)\boldsymbol{G}, \boldsymbol{G}) d\lambda_{\boldsymbol{x}}(\boldsymbol{F}_0, \boldsymbol{G}) \right),$$

and $(\mathsf{T}B_{k,r})(\boldsymbol{x})$ are bounded, and σ is a finite measure. Therefore, by bounded convergence theorem

$$\lim_{k\to\infty} \int_{B_\Omega(\boldsymbol{a},r)} \mathcal{I}_{k,r}(\boldsymbol{x}) d\sigma(\boldsymbol{x}) = \int_{B_\Omega(\boldsymbol{a},r)} \mathcal{I}(\boldsymbol{x}) d\sigma(\boldsymbol{x}),$$

This finishes the proof of Lemma 3.9.

■

By Radon measure version of the Lebesgue differentiation theorem [6, Theorem 2.9.8],

$$\lim_{r\to 0} \frac{1}{\sigma(B_\Omega(\boldsymbol{a}, r))} \int_{B_\Omega(\boldsymbol{a},r)} \mathcal{I}(\boldsymbol{x}) d\sigma(\boldsymbol{x}) = \mathcal{I}(\boldsymbol{a}), \tag{3.49}$$

for $\sigma-$ a.e.$\boldsymbol{a} \in \overline{\Omega}$.

This finishes the proof of Theorem 3.4.

■

3.5 Proof of Theorem 2.2

Recall that we need to consider only the case $\alpha_n \to 0$, as $n \to \infty$. In this case

$$\delta' E(\{\boldsymbol{\phi}_n\}) = \int_{\overline{\Omega}} \mathcal{I}(\boldsymbol{a}) d\sigma(\boldsymbol{a}) + \frac{1}{2} \int_{\overline{\Omega}} \int_{\mathbb{M}} (\mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x}))\boldsymbol{F}, \boldsymbol{F}) d\nu_{\boldsymbol{x}}(\boldsymbol{F}) d\boldsymbol{x}, \tag{3.50}$$

where $\mathcal{I}(\boldsymbol{a})$ is given by (3.29).

To complete the proof we prove both terms in the right-hand side of (3.50) are non-negative.

Step 1. By Theorem 3.1, $\boldsymbol{z}_n \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$. Therefore,

$$\frac{1}{2}\int_\Omega \mathsf{L}(\nabla \boldsymbol{y}_0(\boldsymbol{x})\nabla \boldsymbol{z}_n, \nabla \boldsymbol{z}_n)d\boldsymbol{x} \geq 0,$$

for all n. Taking limit as $n \to \infty$ in the above inequality we prove that the second term on the right-hand side of (3.50) is non-negative. Let us show that the first term is also non- negative.

Step 2. Fix any $\boldsymbol{a} \in \overline{\Omega}$. Then there exists $r(\boldsymbol{a}) > 0$, such that (2.14) is satisfied. Let $\boldsymbol{\phi}_{n,k,r}(\boldsymbol{x}) = \alpha_n \theta_k(\frac{\boldsymbol{x}-\boldsymbol{a}}{r})\boldsymbol{v}_n(\boldsymbol{x})$. Then

$$\mathcal{I}(\boldsymbol{a}) = \lim_{r\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\alpha_n^2\sigma(B_\Omega(\boldsymbol{a},r))}\int_{B_\Omega(\boldsymbol{a},r)} W^0\left(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{\phi}_{n,k,r}(\boldsymbol{x})\right)d\boldsymbol{x}. \quad (3.51)$$

Observe that $\boldsymbol{\phi}_{n,k,r} \in W_0^{1,\infty}(B_\Omega(\boldsymbol{a},r);\mathbb{R}^m)$, because $\boldsymbol{v}_n \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$, and $\theta_k \in C_0^\infty(B(0,1);\mathbb{R}^m)$. The estimate

$$\|\nabla\boldsymbol{\phi}_{n,k,r}\|_2^2 \leq C(k,r)\alpha_n^2(\|\boldsymbol{v}_n\|_2^2 + \|\nabla\boldsymbol{v}_n\|_2^2) \to 0, \text{ as } n\to\infty$$

shows that (2.14) is applicable to $\boldsymbol{\phi}_{n,k,r}$. Therefore,

$$\varliminf_{n\to\infty}\frac{1}{\|\nabla\boldsymbol{\phi}_{n,k,r}\|_2^2}\int_{B_\Omega(a,r)} W^0\left(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{\phi}_{n,k,r}(\boldsymbol{x})\right)d\boldsymbol{x} \geq 0. \quad (3.52)$$

Observe that

$$\frac{\|\nabla\boldsymbol{\phi}_{n,k,r}\|_2^2}{\alpha_n^2} \leq C(k,r)(\|\boldsymbol{v}_n\|_2^2 + \|\nabla\boldsymbol{v}_n\|_2^2) \leq C(k,r).$$

Therefore, for all $k \in \mathbb{N}$, and all $r \in (0, r(\boldsymbol{a}))$

$$\varliminf_{n\to\infty}\frac{1}{\alpha_n^2}\int_{B_\Omega(a,r)} W^0\left(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{\phi}_{n,k,r}(\boldsymbol{x})\right)d\boldsymbol{x} \geq 0. \quad (3.53)$$

The inequality (3.53) is a consequence of (3.52) and the following simple lemma.

Lemma 3.13 *Suppose*

$$\varliminf_{n\to\infty} b_n \geq 0, \quad and \quad 0 \leq |a_n| \leq C.$$

Then

$$\varliminf_{n\to\infty} a_n b_n \geq 0.$$

Proof : If

$$\varliminf_{n\to\infty} a_n b_n = \gamma < 0,$$

then there exists a subsequence n_k such that for all $k \geq 1$, $a_{n_k} b_{n_k} < \frac{\gamma}{2} < 0$. This implies that $b_{n_k} < \frac{\gamma}{2a_{n_k}} \leq \frac{\gamma}{2C}$.

Thus,

$$\varliminf_{n\to\infty} b_n \leq \varliminf_{n\to\infty} b_{n_k} \leq \frac{\gamma}{2C} < 0.$$

Contradiction.

■

Applying the lemma to $a_n = \frac{\|\nabla \boldsymbol{\phi}_{n,k,r}\|_2^2}{\alpha_n^2}$, and

$$b_n = \frac{1}{\|\nabla \boldsymbol{\phi}_{n,k,r}\|_2^2} \int_{B_\Omega(\boldsymbol{a},r)} W^0 \left(\nabla \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{\phi}_{n,k,r}(\boldsymbol{x})\right) d\boldsymbol{x},$$

we obtain (3.53). Hence, $\mathcal{I}(\boldsymbol{a})$ is non-negative as a limit of a sequence of non-neative numbers. Theorem 2.2 is now proved.

CHAPTER 4

HIGHER ORDER REGULARITY

In this chapter we assume that $\boldsymbol{y}_0 \in \mathcal{A}$, satisfies our new sufficient conditions and we prove a global higher regularity result. Our idea, studied more systematically in [10], is that inner variations should be understood as motions of singularities.

Theorem 4.1 *Suppose* $\boldsymbol{y}_0$ *and* W *satisfy all assumptions of Theorem* 2.1. *Then* $\boldsymbol{y}_0 \in W^{2,2}_{loc}(\Omega;\mathbb{R}^m)$. *Moreover,* $h|\nabla\nabla\boldsymbol{y}_0| \in L^2(\Omega)$, *for all* $h \in W^{1,2}_0(\Omega)$.

Let us make an inner variation

$$\boldsymbol{x} \mapsto \boldsymbol{x} + \epsilon\boldsymbol{h}(\boldsymbol{x}), \tag{4.1}$$

where $\boldsymbol{h} \in C_0(\Omega;\mathbb{R}^d) \cap C^1(\overline{\Omega};\mathbb{R}^d)$. What we mean is that instead of $\boldsymbol{y}_0(\boldsymbol{x})$ we consider the competitor $\boldsymbol{y}_\epsilon(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}_\epsilon(\boldsymbol{x}))$, where $\boldsymbol{x}_\epsilon(\boldsymbol{x})$ is the inverse of $\boldsymbol{x} \mapsto \boldsymbol{x} + \epsilon\boldsymbol{h}(\boldsymbol{x})$.

Observe that $\boldsymbol{y}_\epsilon(\boldsymbol{x}) \to \boldsymbol{y}_0(\boldsymbol{x})$ in $C(\overline{\Omega};\mathbb{R}^m)$, as $\epsilon \to 0$.

Therefore the corresponding functional increment

$$\triangle E_\epsilon = \int_\Omega W(\nabla\boldsymbol{y}_0(\boldsymbol{x}_\epsilon(\boldsymbol{x}))\nabla\boldsymbol{x}_\epsilon(\boldsymbol{x}))d\boldsymbol{x} - \int_\Omega W(\nabla\boldsymbol{y}_0(\boldsymbol{x}))d\boldsymbol{x}, \tag{4.2}$$

as a function of ϵ has a local minimum at $\epsilon = 0$. Therefore

$$\frac{d(\triangle E_\epsilon)}{d\epsilon}|_{\epsilon=0} = 0.$$

Notice that we can not differentiate under the integral sign in (4.2), because $\nabla \boldsymbol{y}_0(\boldsymbol{x})$ is not assumed to be smooth. However, differentiation under the integral sign will be possible if we make a change of variables $\boldsymbol{x}' = \boldsymbol{x}_\epsilon(\boldsymbol{x})$ in the first integral of (4.2):

$$\triangle E_\epsilon = \int_\Omega (V(\boldsymbol{x}, \epsilon\nabla \boldsymbol{h}) - V(\boldsymbol{x}, 0))d\boldsymbol{x}, \tag{4.3}$$

where

$$V(\boldsymbol{x}, \boldsymbol{G}) = W(\nabla \boldsymbol{y}_0(\boldsymbol{x})(\boldsymbol{I} + \boldsymbol{G})^{-1}) \det(\boldsymbol{I} + \boldsymbol{G}).$$

The function $V(\boldsymbol{x}, \boldsymbol{G})$ may be discontinuous in $\boldsymbol{x}$, but it is smooth in $\boldsymbol{G}$. Therefore, we can differentiate under the integral sign in (4.3) to obtain

$$0 = \frac{d(\triangle E_\epsilon)}{d\epsilon}|_{\epsilon=0} = \int_\Omega (V_{\boldsymbol{G}}(\boldsymbol{x}, 0), \nabla \boldsymbol{h}(\boldsymbol{x}))d\boldsymbol{x}. \tag{4.4}$$

Notice that due to (4.4)

$$\triangle E_\epsilon = \triangle' E_\epsilon = \int_\Omega \left[V(\boldsymbol{x}, \epsilon\nabla \boldsymbol{h}) - V(\boldsymbol{x}, 0) - \epsilon(V_{\boldsymbol{G}}(\boldsymbol{x}, 0), \nabla \boldsymbol{h}(\boldsymbol{x}))\right] d\boldsymbol{x}.$$

Lemma 4.1 *There exists a constant $C > 0$ such that for all $\boldsymbol{h} \in C_0(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$*

$$\overline{\lim_{\epsilon\to 0}} \frac{|\triangle' E_\epsilon|}{\|\epsilon\nabla \boldsymbol{h}\|_2^2} \le K.$$

Proof :

From Taylor's expansion of V in $\boldsymbol{G}$ around $(\boldsymbol{x}, 0)$ we have

$$|V(\boldsymbol{x}, \boldsymbol{G}) - V(\boldsymbol{x}, 0) - (V_{\boldsymbol{G}}(\boldsymbol{x}, 0), \boldsymbol{G})| \le K|\boldsymbol{G}|^2,$$

for some $K > 0$, when $|\boldsymbol{G}| \le 1/2$.
But for each $\boldsymbol{h} \in C_0(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ there exists ϵ_0 such that

$$\|\epsilon\nabla \boldsymbol{h}\|_\infty \le \frac{1}{2}, \text{ for all } \epsilon \in (0, \epsilon_0).$$

Hence

$$|\triangle' E_\epsilon| \leq K\epsilon^2 \|\nabla \boldsymbol{h}\|_2^2, \text{ for all } \epsilon < \epsilon_0(\boldsymbol{h}).$$

■

Let

$$\boldsymbol{\phi}_\epsilon(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}_\epsilon(\boldsymbol{x})) - \boldsymbol{y}_0(\boldsymbol{x})$$

be the outer variation corresponding to the inner variation (4.1).

Observe that $\boldsymbol{\phi}_\epsilon(\boldsymbol{x})$ converges to 0 in $C(\overline{\Omega}; \mathbb{R}^m)$, since $\boldsymbol{x}_\epsilon(\boldsymbol{x}) \to \boldsymbol{x}$ uniformly as $\epsilon \to 0$ and $\boldsymbol{y}_0$ is continuous.

By Theorem 2.1

$$\underline{\lim}_{n\to\infty} \frac{\triangle E(\boldsymbol{\phi}_n)}{\|\nabla \boldsymbol{\phi}_n\|_2^2} \geq \beta > 0. \tag{4.5}$$

Applying Lemmas 4.1 and inequality (4.5) to $\boldsymbol{\phi}_\epsilon$, we obtain

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla \boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla \boldsymbol{h}\|_2^2} = \overline{\lim_{\epsilon\to 0}} \frac{\frac{|\Delta' E_\epsilon|}{\|\epsilon\nabla \boldsymbol{h}\|_2^2}}{\frac{|\Delta E(\boldsymbol{\phi}_\epsilon)|}{\|\nabla \boldsymbol{\phi}_\epsilon\|_2^2}} \leq \frac{\overline{\lim}_{\epsilon\to 0} \frac{|\Delta' E_\epsilon|}{\|\epsilon\nabla \boldsymbol{h}\|_2^2}}{\underline{\lim}_{\epsilon\to 0} \frac{|\Delta E(\boldsymbol{\phi}_\epsilon)|}{\|\nabla \boldsymbol{\phi}_\epsilon\|_2^2}} \leq \frac{K}{\beta}, \tag{4.6}$$

for all $\boldsymbol{h} \in C_0(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$.

Lemma 4.2 *There exists a constant $C > 0$ so that for all $\boldsymbol{h} \in C_0(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$*

$$\overline{\lim_{\epsilon\to 0}} \frac{\int_\Omega |\nabla \boldsymbol{y}_0(\boldsymbol{x}) - \nabla \boldsymbol{y}_0(\boldsymbol{x} + \epsilon \boldsymbol{h}(\boldsymbol{x}))|^2 d\boldsymbol{x}}{\int_\Omega |\epsilon \nabla \boldsymbol{h}|^2 d\boldsymbol{x}} \leq C. \tag{4.7}$$

Proof : Observe that

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla \boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla \boldsymbol{h}\|_2^2} = \overline{\lim_{\epsilon\to 0}} \frac{\int_\Omega |\nabla \boldsymbol{y}_0(\boldsymbol{x}_\epsilon(\boldsymbol{x}))(\boldsymbol{I} + \epsilon \nabla \boldsymbol{h}(\boldsymbol{x}_\epsilon(\boldsymbol{x})))^{-1} - \nabla \boldsymbol{y}_0(\boldsymbol{x})|^2 d\boldsymbol{x}}{\int_\Omega |\epsilon \nabla \boldsymbol{h}|^2 d\boldsymbol{x}} \tag{4.8}$$

Making change of variables $\boldsymbol{x}' = \boldsymbol{x}_\epsilon(\boldsymbol{x})$ in (4.8), we get

$$\overline{\lim_{\epsilon\to 0}} \frac{\|\nabla \boldsymbol{\phi}_\epsilon\|_2^2}{\|\epsilon\nabla \boldsymbol{h}\|_2^2} =$$
$$\overline{\lim_{\epsilon\to 0}} \frac{\int_\Omega |\nabla \boldsymbol{y}_0(\boldsymbol{x}')(\boldsymbol{I} + \epsilon \nabla \boldsymbol{h})^{-1} - \nabla \boldsymbol{y}_0(\boldsymbol{x}' + \epsilon \boldsymbol{h}(\boldsymbol{x}'))|^2 \det(\boldsymbol{I} + \epsilon \nabla \boldsymbol{h}) d\boldsymbol{x}'}{\int_\Omega |\epsilon \nabla \boldsymbol{h}|^2 d\boldsymbol{x}}.$$

And $\det(\boldsymbol{I} + \epsilon \nabla \boldsymbol{h}) \geq \frac{1}{2}$, when ϵ is small enough, since $\det(\boldsymbol{I} + \epsilon \nabla \boldsymbol{h}) \to 1$, uniformly as $\epsilon \to 0$.

Thus

$$\overline{\lim_{\epsilon\to 0}}\frac{\int_\Omega |\nabla \boldsymbol{y}_0(\boldsymbol{x}')(\boldsymbol{I}+\epsilon\nabla\boldsymbol{h})^{-1} - \nabla\boldsymbol{y}_0(\boldsymbol{x}'+\epsilon\boldsymbol{h}(\boldsymbol{x}'))|^2 d\boldsymbol{x}'}{\int_\Omega |\epsilon\nabla\boldsymbol{h}|^2 d\boldsymbol{x}} \le \frac{2K}{\beta}.$$

Observe that for ϵ small enough $|(\boldsymbol{I}+\epsilon\nabla\boldsymbol{h})^{-1}| > 1/2$. Therefore,

$$|\nabla\boldsymbol{y}_0(\boldsymbol{x})(\boldsymbol{I}+\epsilon\nabla\boldsymbol{h})^{-1} - \nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))| \ge \frac{1}{2}|\nabla\boldsymbol{y}_0(\boldsymbol{x}) - \nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))(\boldsymbol{I}+\epsilon\nabla\boldsymbol{h})|. \tag{4.9}$$

This inequality is a corollary of the following lemma.

Lemma 4.3 *Let σ_{min} and σ_{max} be the minimal and maximal singular values of a $d\times d$ matrix $\mathbf{A}$, respectively. Then*

$$\sigma_{min}|\mathbf{B}| \le |\mathbf{BA}| \le \sigma_{max}|\mathbf{B}|$$

for all $m\times d$ matrices $\mathbf{B}$.

Proof :

$|\mathbf{BA}|^2 = \mathrm{Tr}(\mathbf{AA}^t\mathbf{B}^t\mathbf{B})$. Observe that $\mathbf{AA}^t \ge \sigma^2_{\min}\boldsymbol{I}$ and

$$|\mathbf{BA}|^2 = \mathrm{Tr}((\mathbf{AA}^t - \sigma^2_{\min}\boldsymbol{I})\mathbf{B}^t\mathbf{B}) + \sigma^2_{\min}|\mathbf{B}|^2. \tag{4.10}$$

By a theorem of Schur (see e.g.[15, Theorem 10.7]), the first term on the right-hand side of (4.10) is non-negative, since the matrices $\mathbf{AA}^t - \sigma^2_{\min}\boldsymbol{I}$ and $\mathbf{B}^t\mathbf{B}$ are symmetric and non-negative definite. Similarly,

$$|\mathbf{BA}|^2 = \sigma^2_{\max}|\mathbf{B}|^2 - \mathrm{Tr}((\sigma^2_{\max}\boldsymbol{I} - \mathbf{AA}^t)\mathbf{B}^t\mathbf{B}) \le \sigma^2_{\max}|\mathbf{B}|^2.$$

■

The inequality

$$|\nabla\boldsymbol{y}_0(\boldsymbol{x}) - \nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))|^2 \le$$
$$2|\nabla\boldsymbol{y}_0(\boldsymbol{x}) - \nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})) - \nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))\epsilon\nabla\boldsymbol{h}|^2 +$$
$$2\epsilon^2|\nabla\boldsymbol{y}_0(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))\nabla\boldsymbol{h}|^2$$

together with (4.9) implies (4.7). This completes the proof of Lemma 4.2.

■

The following Lemma applied to every component of the matrix field $\nabla\boldsymbol{y}_0(\boldsymbol{x})$ finishes the proof of Theorem 4.1.

Lemma 4.4 *Let Ω be an open bounded domain in $\mathbb{R}^d$. Let $f \in L^\infty(\Omega)$ be such that for all $\boldsymbol{h} \in C_0(\Omega;\mathbb{R}^d) \cap C^1(\overline{\Omega};\mathbb{R}^d)$*

$$\overline{\lim_{\epsilon\to 0}} \frac{1}{\epsilon^2} \int_\Omega |f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})) - f(\boldsymbol{x})|^2 d\boldsymbol{x} \le C \int_\Omega |\nabla \boldsymbol{h}(\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{4.11}$$

Then $f \in W^{1,2}_{loc}(\Omega)$ and $h\nabla f \in L^2(\Omega)$, for all $h \in W^{1,2}_0(\Omega)$.

Proof :

In view of (4.11) there exists a subsequence, not relabeled, and a function g (both dependent on $\boldsymbol{h}$) such that

$$\frac{f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x})) - f(\boldsymbol{x})}{\epsilon} \rightharpoonup g$$

weakly in $L^2(\Omega)$. In particular

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_\Omega f(\boldsymbol{x}+\epsilon\boldsymbol{h}(\boldsymbol{x}))\phi(\boldsymbol{x})d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}\right] = \int_\Omega g(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}, \tag{4.12}$$

for all $\phi \in C_0^\infty(\Omega)$. Making change of variables $\boldsymbol{x}' = \boldsymbol{x} + \epsilon\boldsymbol{h}(\boldsymbol{x})$ in the first integral of (4.12), and using the fact that $\boldsymbol{x}_\epsilon(\boldsymbol{x}) \to \boldsymbol{x}$ in $C^1(\overline{\Omega};\mathbb{R}^d)$, we get

$$\int_\Omega g\phi d\boldsymbol{x} = \lim_{\epsilon\to 0}\int_\Omega f(\boldsymbol{x}')\left(\frac{\phi(\boldsymbol{x}_\epsilon(\boldsymbol{x}'))\det(\nabla\boldsymbol{x}_\epsilon(\boldsymbol{x}')) - \phi(\boldsymbol{x}')}{\epsilon}\right)d\boldsymbol{x}'$$

$= -\int_\Omega f\nabla\cdot(\phi\boldsymbol{h})d\boldsymbol{x}'$.

It follows that $\nabla\cdot(f\boldsymbol{h}) = g + f\nabla\cdot\boldsymbol{h}$ in the sense of distributions.

Now let $\boldsymbol{h}(\boldsymbol{x}) = h(\boldsymbol{x})\boldsymbol{e}_i$ for some $h \in C_0(\Omega)\cap C^1(\overline{\Omega})$, where $\boldsymbol{e}_i$ is the i^{th} standard basis vector. Then

$$\frac{\partial}{\partial x_i}(f(\boldsymbol{x})h(\boldsymbol{x})) = \nabla\cdot(f(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})) = g + f\frac{\partial h}{\partial \boldsymbol{x}_i} \in L^2(\Omega).$$

This implies that $fh \in W^{1,2}(\Omega)$, and therefore $f \in W^{1,2}_{loc}(\Omega)$. Thus, it follows from (4.11) that

$$\int_\Omega (\nabla f, \boldsymbol{h})^2 d\boldsymbol{x} \le C\|\nabla\boldsymbol{h}\|_2^2 \tag{4.13}$$

for all $\boldsymbol{h} \in C_0(\Omega;\mathbb{R}^d)\cap C^1(\overline{\Omega};\mathbb{R}^d)$.

In order to prove the last claim in the Lemma, we fix $\boldsymbol{h} \in W_0^{1,2}(\Omega;\mathbb{R}^d)$ and consider a sequence $\{\boldsymbol{h}_n\colon n \ge 1\} \subset C_0^\infty(\Omega;\mathbb{R}^d)$, such that $\boldsymbol{h}_n \to \boldsymbol{h}$ in

$W_0^{1,2}(\Omega;\mathbb{R}^d)$. It follows that there is a subsequence (not relabeled) such that $\boldsymbol{h}_n \to \boldsymbol{h}$ for almost every $\boldsymbol{x} \in \Omega$. By Fatou's lemma

$$\int_\Omega (\nabla f(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x}))^2 d\boldsymbol{x} \leq \varliminf_{n\to\infty} \int_\Omega (\nabla f(\boldsymbol{x}), \boldsymbol{h}_n(\boldsymbol{x}))^2 d\boldsymbol{x} \leq C\|\nabla \boldsymbol{h}\|_2^2.$$

Taking $\boldsymbol{h}(\boldsymbol{x}) = h(\boldsymbol{x})\boldsymbol{e}_i$ finishes the proof of the Lemma.

■

Theorem 4.1 is now proved. ■

REFERENCES CITED

[1] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, 63(4):337–403, 1976/77.

[2] J. M. Ball and F. Murat. $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.*, 58(3):225–253, 1984.

[3] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.*, 108(4):667–689, 1987.

[4] L. C. Evans. Quasiconvexity and partial regularity in the calculus of variations. *Arch. Ration. Mech. Anal.*, 95(3):227–252, 1986.

[5] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.

[6] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[7] I. Fonseca, S. Müller, and P. Pedregal. Analysis of concentration and oscillation effects generated by gradients. *SIAM J. Math. Anal.*, 29(3):736–756, 1998.

[8] Y. Grabovsky and T. Mengesha. Direct approach to the problem of strong local minima in calculus of variations. *Calc. Var. and PDE*, 29:59–83, 2007.

[9] Y. Grabovsky and T. Mengesha. Sufficient conditions for strong local minima: the case of C^1 extremals. *Trans. Amer. Math. Soc.*, To appear. http://www.math.temple.edu/~yury/strong-loc-min.pdf.

[10] Y. Grabovsky and L. Truskinovsky. Metastability in nonlinear elsticity. In preparation.

[11] J. N. Hao W., S. Leonardi. An example of irregular solution to a nonlinear euler-lagrange elliptic system with real analytic coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (4) 23, no. 1:57–67, 1996.

[12] F. Hüsseinov. Weierstrass condition for the general basic variational problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 125(4):801–806, 1995.

[13] J. Kristensen. Finite functionals and young measures generated by gradients of sobolev functions. Technical Report Mat-Report No. 1994-34, Mathematical Institute, Technical University of Denmark, 1994.

[14] J. Kristensen and A. Taheri. Partial regularity of strong local minimizers in the multi-dimensional calculus of variations. *Arch. Ration. Mech. Anal.*, 170(1):63–89, 2003.

[15] P. D. Lax. *Linear algebra.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1997. A Wiley-Interscience Publication.

[16] J. Morrey, Charles B. *Multiple integrals in the calculus of variations.* Springer-Verlag New York, Inc., New York, 1966. Die Grundlehren der mathematischen Wissenschaften, Band 130.

[17] Nečas. Example of an irregular solution to a nonlinear elliptic sysytem with analytic coefficients and conditions of regularity, theory of non linear

operators, abhandlungen akad, derwissen. der ddr. *Proc. of a Summer School held in Berlin (1975)*, 1997.

[18] P. Pedregal. *Parametrized measures and variational principles.* Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser Verlag, Basel, 1997.

[19] V. Šverák and X. Yan. A singular minimizer of a smooth strongly convex functional in three dimensions. *Calc. Var. Partial Differential Equations*, 10(3):213–221, 2000.