On Boundary Values of Solutions in Involutive Structures

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ABSTRACT

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An involutive structure is a pair $(M, \mathcal{V})$ where M is a C^∞ manifold and $\mathcal{V}$ is a subbundle of the complexified tangent bundle $\mathbb{C}TM$ which is involutive, that is, the bracket of two smooth sections of $\mathcal{V}$ is also a smooth section of $\mathcal{V}$. The involutive structure $(M, \mathcal{V})$ is called locally integrable if the orthogonal of $\mathcal{V}$ in $\mathbb{C}T^*M$ is locally generated by exact forms. In Chapter 1, we will study hypo-analytic structures which are special locally integrable structures. A microlocal theory of hypo-analyticity was developed in [BCT] and it was used to describe the regularity of solutions in [BCT]. A more invariant definition of microlocal hypo-analyticity was given more recently by Eastwood and Graham [EG]. We will present a proof of the equivalence of the notions of microlocal hypo-analyticity given in the works [BCT] and [EG]. We will then use the definition of microlocal hypo-analyticity given in [EG] to present a proof of a criterion (see Theorem 34) for a distribution u on a maximally real submanifold X in $\mathbb{C}^m$ to be expressible as the sum of boundary values of holomorphic functions on prescribed wedges. The hypo-analytic wave-front set of u, $WF^X(u)$, is constrained as a consequence of the fact that u extends as a holomorphic function to a wedge. We then prove a result (see Theorem 42) which shows how to decompose a distribution u on a maximally real submanifold in $\mathbb{C}^m$ as a sum of distributions u_j, $1 \leq j \leq N$, whose hypo-analytic wavefront sets are contained in pre-assigned cones.

In Chapter 2, we study existence of boundary values of solutions defined on wedges; this can be summarized as follows: Let N be a submanifold of a smooth manifold M. In a neighborhood of a point of N we may introduce coordinates (x', x'') for M with $x' \in \mathbb{R}^r$ and $x'' \in \mathbb{R}^s$ in which, locally, $N = \{x'' = 0\}$. By a wedge in M with edge N we mean an open set $\mathcal{W} \subseteq M$ which in some such coordinate system is of the form $\mathcal{W} = \mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is a ball in $\mathbb{R}^r$ and $\mathcal{C}$ is a truncated, open convex cone in $\mathbb{R}^s \backslash \{0\}$. When $(M, \mathcal{V})$

is a hypo-analytic structure, a submanifold E of M is called strongly noncharacteristic if $\mathbb{C}T_pM = \mathbb{C}T_pE + \mathcal{V}_p$ for each $p \in E$, and maximally real if $\mathbb{C}T_pM = \mathbb{C}T_pE \oplus \mathcal{V}_p$ for each $p \in E$. Suppose $\mathcal{W}$ is a wedge in M whose edge E is maximally real. Let $u \in \mathcal{D}'(\mathcal{W})$ be a solution of $\mathcal{V}$. Let (x', x'') be a coordinate system in which $E = \{x'' = 0\}$ and $\mathcal{W} = \mathcal{B} \times \mathcal{C}$ as above. It is known that the solution u is a smooth function of $x'' \in \mathcal{C}$ valued in distributions in x'-space $\mathcal{B}$. We will prove (see Theorem 45) a sufficient condition for the existence of a boundary value for u, bu, at $x'' = 0$ when u is continuous on the wedge $\mathcal{W}$. This generalizes previous results in [BH1] and [BH2]. Then we prove a similar result (see Theorem 50) when our involutive structure is not necessarily locally integrable.

In Chapter 3, we study Edge-of-the-Wedge theory in involutive structures that are not necessarily locally integrable (see Theorems 58 and 61).

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Microlocal Hypo-analyticity and the FBI Transform

1.1 Introduction

In this chapter, we study microlocal regularity properties of the distributions u on a maximally real submanifold X of a hypo-analytic manifold M that arise as the boundary values of holomorphic functions on wedges in M with edge X. The hypo-analytic wave-front set of u is constrained as a consequence of the fact that u extends as a holomorphic function to a wedge.

1.2 Hypo-analytic Structures

Definition 1 *Let $(M, \mathcal{V})$ be a locally integrable structure, where $\dim_{\mathbb{R}} M = m + n$, and $\dim_{\mathbb{C}} \mathcal{V} = n$. Suppose that M can be covered by charts (U_α, Z_α), where $U_\alpha \subset M$ is open and $Z_\alpha = \left(Z^1_\alpha, ..., Z^m_\alpha\right) : U_\alpha \to \mathbb{C}^m$ are a complete set of first integrals; (i.e., $dZ^1_\alpha, ..., dZ^m_\alpha$ are everywhere linearly independent and $\mathcal{V} Z_\alpha = 0$). Suppose further that whenever $U_\alpha \cap U_\beta \neq \varnothing$, there exists a local biholomorphism*

$$f_{\alpha\beta} : U \overset{open}{\subset} \mathbb{C}^m \to \mathbb{C}^m$$

such that

$$f_{\alpha\beta} \circ \left(Z_\alpha|_{U_\alpha \cap U_\beta}\right) = Z_\beta|_{U_\alpha \cap U_\beta}.$$

Then we say that $(M, \mathcal{V})$ is a hypo-analytic manifold. Here, the number m is called the dimension of the hypo-analytic structure, and n its codimension.

Definition 2 *A function $f : M \to \mathbb{C}$ on a hypo-analytic manifold M is said to be hypo-analytic if in a neighborhood of each point $p \in M$ it is of the form*

$$f = h(Z_1, ..., Z_m)$$

where h is holomorphic and defined in a neighborhood of $(Z_1(p), ..., Z_m(p))$ in $\mathbb{C}^m$.

In other words, f is hypo-analytic at $p \in M$ if f can be represented by a convergent power series in $(Z_1, ..., Z_m)$ in some neighborhood of p in M.

Definition 3 *We define the structure bundle T' of M by*

$$T' = \bigcup_{p \in M} T'_p,$$

where

$$T'_p = \{\omega \in \mathbb{C}T^*_p M : \langle \omega, v \rangle = 0 \text{ for all } v \in \mathcal{V}_p\} = span_{\mathbb{C}}\{dZ_1(p), ..., dZ_m(p)\}.$$

Definition 4 *Let $(M, \mathcal{V})$ be involutive. A submanifold $X \subset M$ is called maximally real if the pullback map*

$$\pi^* : \mathbb{C}T^*M|_X \to \mathbb{C}T^*X$$

induces an isomorphism

$$T'|_X \cong \mathbb{C}T^*X.$$

Note that, therefore, $\dim_{\mathbb{R}} X = m$. The next lemma gives other equivalent definitions of maximally real submanifolds.

Lemma 5 *Let $X \subset M$ be a submanifold. Then the following are equivalent:*

(i) X is maximally real;

(ii) $\mathbb{C}T_pM = \mathbb{C}T_pX \oplus \mathcal{V}_p$ for all $p \in X$; and

(iii) $\mathbb{C}T^*_pM = \mathbb{C}N^*_pX \oplus T'_p$ for all $p \in X$.

Proof. *(i)* $\Longrightarrow$ *(ii)* Suppose that $X \subset M$ is maximally real. This means that the pullback map $\pi^* : \mathbb{C}T^*M|_X \to \mathbb{C}T^*X$ induces an isomorphism $T'|_X \cong \mathbb{C}T^*X$. Let $p \in X$. If $\{\omega_1, ..., \omega_m\}$ is a basis of T'_p, then $\{\pi^*(\omega_1), ..., \pi^*(\omega_m)\}$ is a basis of $\mathbb{C}T^*_pX$. Let $v \in \mathbb{C}T_pX \cap \mathcal{V}_p$.

Being in $\mathcal{V}_p$, we have that $\langle \pi^*(\omega_j), v\rangle = 0$ for all $1 \leq j \leq m$. Thus, $\left\langle \mathbb{C}T_p^*X, v\right\rangle = 0$ and since $v \in \mathbb{C}T_pX$, $v = 0$. Hence, $\mathbb{C}T_pX \cap \mathcal{V}_p = \{0\}$. Since $\mathbb{C}T_pX \oplus \mathcal{V}_p \subseteq \mathbb{C}T_pM$, $\dim_{\mathbb{C}} \mathbb{C}T_pX = m$, $\dim_{\mathbb{C}} \mathcal{V} = n$, and $\dim_{\mathbb{C}} \mathbb{C}T_pM = m + n$, we get that $\mathbb{C}T_pM = \mathbb{C}T_pX \oplus \mathcal{V}_p$.

(ii) $\Longrightarrow$ *(iii)* Fix $p \in X$ and let $\omega \in \mathbb{C}N_p^*X \cap T_p' \subseteq \mathbb{C}T_p^*M$. Then $\langle \omega, \mathbb{C}T_pX\rangle = 0$ and $\langle \omega, \mathcal{V}_p\rangle = 0$. But $\mathbb{C}T_pM = \mathbb{C}T_pX \oplus \mathcal{V}_p$. Hence, $\langle \omega, \mathbb{C}T_pM\rangle = 0$ and so $\omega = 0$. Hence, $\mathbb{C}N_p^*X \cap T_p' = \{0\}$. Since $\mathbb{C}N_p^*X \oplus T_p' \subseteq \mathbb{C}T_p^*M$, $\dim_{\mathbb{C}} \mathbb{C}N_p^*X = n$, $\dim_{\mathbb{C}} T_p' = m$, and $\dim_{\mathbb{C}} \mathbb{C}T_p^*M = m + n$, we get that $\mathbb{C}T_p^*M = \mathbb{C}N_p^*X \oplus T_p'$.

(iii) $\Longrightarrow$ *(i)* We need to show that the pullback map $\pi^* : \mathbb{C}T^*M|_X \to \mathbb{C}T^*X$ induces an isomorphism $T'|_X \cong \mathbb{C}T^*X$. Since $\dim_{\mathbb{C}} T_p' = m = \dim_{\mathbb{C}} \mathbb{C}T_p^*X$, it suffices to show that $\pi^* : T_p' \to \mathbb{C}T_p^*X$ is injective for every $p \in X$. So, fix $p \in X$ and let $\omega \in T_p'$. Suppose that $\pi^*(\omega) = 0$. Then $0 = \langle \pi^*(\omega), \mathbb{C}T_pX\rangle = \langle \omega, \mathbb{C}T_pX\rangle$. Hence, $\omega \in \mathbb{C}N_p^*X$. Thus, $\omega \in \mathbb{C}N_p^*X \cap T_p' = \{0\}$ and so $\omega = 0$. This shows that $\pi^* : T_p' \to \mathbb{C}T_p^*X$ is injective and hence, an isomorphism. ■

Definition 6 *Let $X \subset M$ be maximally real. The real structure bundle of X, denoted by $\mathbb{R}T_X'$, is the image of the real cotangent bundle of X, T^*X, under the natural isomorphism $T'|_X \cong \mathbb{C}T^*X$.*

Definition 7 *The characteristic set of M, denoted T^0, is defined to be*

$$T^0 = T' \cap T^*M.$$

It can be easily shown that if $X \subset M$ is a maximally real submanifold, then $T^0|_X \subset \mathbb{R}T_X'$.

Suppose that $(M, \mathcal{V})$ is a hypo-analytic manifold, $X \subset M$ is maximally real, $p \in X$, and let $\{Z_1, ..., Z_m\}$ be a complete set of first integrals near p in M. Then we have that $\{d(Z_j|_X) : 1 \leq j \leq m\}$ is a basis of $\mathbb{C}T^*X$. Since $\mathcal{V}_pX = \mathcal{V}_p \cap \mathbb{C}T_pX = \{0\}$, X inherits a hypo-analytic structure from M of codimension 0.

From the *Baouendi-Treves Approximation Formula*, we get the following result:

Proposition 8 *If $(M, \mathcal{V})$ is locally integrable, $X \subset M$ is maximally real, and f is a solution such that $f|_X = 0$, then $f \equiv 0$ in a neighborhood of X in M.*

As an immediate consequence, one has the following proposition:

Proposition 9 *Suppose* $(M, \mathcal{V})$ *is a hypo-analytic manifold,* $X \subset M$ *maximally real, and* h *a solution in a neighborhood of* X *in* M*. Let* $p_0 \in X$ *and* $Z = (Z_1, ..., Z_m)$ *be a complete set of first integrals near* p_0*. Suppose further that* H *is holomorphic near* $Z(p_0)$ *and* $h(x) = H(Z(x))$ *for* $x \in X$ *near* p_0*. Then* $h(p) = H(Z(p))$ *for* $p \in M$ *near* p_0*.*

Hence, to study regularity (hypo-analyticity) of a solution h, it is enough to study the restriction $h|_X$ where $X \subset M$ is maximally real.

Now, let X be a manifold with a hypo-analytic structure of codimension 0, (such an X will often arise as a maximally real submanifold of a large hypo-analytic manifold), and let $p \in X$. We may choose our hypo-analytic chart Z such that $Z(p) = 0$ and $\operatorname{Im} dZ(p) = 0$, in which case we may take $x_j = \operatorname{Re} Z_j$ $(1 \leq j \leq m)$ as local coordinates on X near p. These coordinates enable us to identify a neighborhood of p in X with a neighborhood of 0 in $\mathbb{R}^m$ and T_p^*X with $T_0^*\mathbb{R}^m \cong \mathbb{R}^m$. Set $\Phi = \operatorname{Im} Z$ so that near 0 in $\mathbb{R}^m$, $Z(x) = x + i\Phi(x) \in \mathbb{C}^m$, where $\Phi(0) = 0,$ and $D\Phi(0) = 0$. Then $Z : X \to Z(X)$ is an embedding near p of X onto a totally real submanifold of $\mathbb{C}^m$ of maximal dimension. We will often identify X with $Z(X)$.

Remark 10 *(Description of the real structure bundle* $\mathbb{R}T'_X$ *near* 0*) Let* $X \subset \mathbb{C}^m$ *be a maximally real submanifold. After a translation and a* $\mathbb{C}$*-linear transformation in* $\mathbb{C}^m$*, we may assume that* $0 \in X$ *and that* $T_0X = \mathbb{R}^m$*. Then in a small enough neighborhood* Ω *of* 0 *in* X*,* Ω *is the image of some open neighborhood* U *of* 0 *in* $\mathbb{R}^m$ *under the map* $x \to Z(x)$ *with* $Z(x) = x + i\Phi(x)$*; where* $\Phi : U \to \mathbb{R}^m$ *is* C^∞*,* $\Phi(0) = 0,$ *and* $\Phi_x(0) = 0$*. Then a point* $(z, \zeta) \in \mathbb{R}T'_X$*, with* $z \in Z(U)$*, if there is* $x \in U$ *and* $\xi \in \mathbb{R}^m$ *such that*

$$z = Z(x) \quad \text{and} \quad \zeta = {}^tZ_x(x)^{-1}\xi.$$

1.3 FBI Transform in a Maximally Real Submanifold of $\mathbb{C}^m$

The variable point in $\mathbb{C}^m$ will be denoted by z or z'; "dual" coordinates will be ζ_j $(1 \leq j \leq m)$. For any number $\tau > 0$ we write

$$\mathcal{C}_\tau = \{\zeta \in \mathbb{C}^m : |\operatorname{Im} \zeta| < \tau\, |\operatorname{Re} \zeta|\}.$$

For any $z = (z_1, ..., z_m) \in \mathbb{C}^m$, we write

$$\langle z \rangle^2 = z \cdot z = z_1^2 + ... + z_m^2$$

and for any $\zeta \in \mathcal{C}_1$, we write

$$\langle\zeta\rangle = (\zeta\cdot\zeta)^{1/2} \quad \text{(main branch of square root)}.$$

Note that $\operatorname{Re}\langle\zeta\rangle^2 > 0$ for all $\zeta \in \mathcal{C}_1$. We shall also use the notation

$$\triangle(z,\zeta) = \det\left(I + i\,(z\odot\zeta)\,/\,\langle\zeta\rangle\right),$$

where $z\odot\zeta$ denotes the $m\times m$ matrix $\left(z_i\zeta_j\right)_{1\le i,j\le m}$; $\triangle(z,\zeta)$ is just the Jacobian determinant of the map

$$\zeta \to \zeta + i\,\langle\zeta\rangle\, z \quad (z\in\mathbb{C}^m,\ \zeta\in\mathcal{C}_1).$$

From now on, let $(M,\mathcal{V})$ be $\mathbb{C}^m$ with the standard complex structure

$$\mathcal{V} = span_{\mathbb{C}}\left\{\frac{\partial}{\partial\overline{z}_j} : 1\le j\le m\right\}.$$

Also, let $X\subset\mathbb{C}^m$ be a maximally real submanifold.

Definition 11 *Let $u\in\mathcal{E}'(X)$. For $(z,\zeta)\in\mathbb{C}^m\times\mathcal{C}_1$, the duality bracket*

$$\mathcal{F}_u(z,\zeta) = \int\limits_X e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2} u(z')\,\triangle\,(z-z',\zeta)dz'$$

will be called the FBI transform of u.

Proposition 12 $\mathcal{F}_u(z,\zeta)\in\mathcal{O}\left(\mathbb{C}^m\times\mathcal{C}_1\right)$.

Proof. Let M_i, $1\le i\le m$, be the vector fields on X defined by the relations $M_i(z_j|_X)=\delta_{ij}$. Then $\{M_1,...,M_m\}$ form a basis of $\mathbb{C}TX$. The structure theorem for compactly supported distributions $u\in\mathcal{E}'(X)$ states that we may write

$$u = \sum_{|\alpha|\le r} M^\alpha u_\alpha \quad (\alpha=(\alpha_1,...,\alpha_m)\in\mathbb{Z}_+^m;\ r\in\mathbb{Z}_+;\ M^\alpha = M_1^{\alpha_1}\cdots M_m^{\alpha_m}),$$

where for each α, u_α is continuous on X and $\operatorname{supp}(u_\alpha)$ is compact and contained in an arbitrary neighborhood of $\operatorname{supp}(u)$. By linearity, one has

$$\mathcal{F}_u(z,\zeta) = \sum_{|\alpha|\le r}\mathcal{F}_{M^\alpha u_\alpha}(z,\zeta).$$

Integration by parts gives

$$\mathcal{F}_{M^\alpha u_\alpha}(z,\zeta) = \int\limits_X e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2} u_\alpha(z')\mathcal{P}_\alpha(z-z',\zeta)dz',$$

where

$$\mathcal{P}_\alpha(z,\zeta) = e^{-i\zeta\cdot z+\langle\zeta\rangle\langle z\rangle^2} M^\alpha\{\triangle(z,\zeta)e^{i\zeta\cdot z-\langle\zeta\rangle\langle z\rangle^2}\} = e^{-i\zeta\cdot z+\langle\zeta\rangle\langle z\rangle^2}\left(\frac{\partial}{\partial z}\right)^\alpha\{\triangle(z,\zeta)e^{i\zeta\cdot z-\langle\zeta\rangle\langle z\rangle^2}\}.$$

To every compact set $K \subset \mathbb{C}^m$ there exists a constant $C_K > 0$ such that

$$|\mathcal{P}_\alpha(z,\zeta)| \leq C_K\,(1+|\zeta|)^{|\alpha|} \quad \text{for all } (z,\zeta) \in K \times \mathcal{C}_1$$

Also, we have

$$\mathcal{F}_{M^\alpha u} = \left(\frac{\partial}{\partial z}\right)^\alpha \mathcal{F}_u,$$

and hence,

$$\mathcal{F}_u(z,\zeta) = \sum_{|\alpha|\leq r}\left(\frac{\partial}{\partial z}\right)^\alpha \mathcal{F}_{u_\alpha}(z,\zeta) = \sum_{|\alpha|\leq r}\int_X e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2} u_\alpha(z')\mathcal{P}_\alpha(z-z',\zeta)dz'.$$

We note that

$$\int_X e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2} u_\alpha(z')\mathcal{P}_\alpha(z-z',\zeta)dz'$$

defines a holomorphic function of $(z,\zeta) \in \mathbb{C}^m \times \mathcal{C}_1$ (This follows since u_α is continuous, and so we can differentiate under the integral sign). ■

Definition 13 *Let $X \subset \mathbb{C}^m$ be a maximally real submanifold and let $z_0 \in X$. We say that X is well-positioned at z_0 if there is a number τ, $0 < \tau < 1$, and an open neighborhood Ω of z_0 in X such that the following is true:*

$$\text{Whatever } z, z' \in \Omega \text{ and } \zeta \in (\mathbb{R}T'_X|_z) \cup (\mathbb{R}T'_X|_{z'}),$$
$$|\operatorname{Im}\zeta| < \tau\,|\operatorname{Re}\zeta|\,;$$
$$\operatorname{Im}\left[\zeta\cdot(z-z') + i\,\langle\zeta\rangle\,\langle z-z'\rangle^2\right] \geq (1-\tau)\,|\zeta|\,|z-z'|^2\,.$$

We shall say that X is *very well-positioned* at z_0 if, given any number τ, $0 < \tau < 1$, there is an open neighborhood Ω of z_0 in X such that the same as above holds.

Proposition 14 *(Proposition IX.2.2 in [T]) Given any maximally real submanifold $X \subset \mathbb{C}^m$, and any point $z_0 \in X$, there exists a biholomorphism H of an open neighborhood O of z_0 in $\mathbb{C}^m$ onto an open neighborhood of the origin, with $H(z_0) = 0$, such that*

$H(X \cap O)$ is very well-positioned at 0.

The following proposition follows easily from the above discussion:

Proposition 15 *Let $X \subset \mathbb{C}^m$ be a maximally real submanifold that is well-positioned at z_0. Then there exists a neighborhood Ω of z_0 in X with the following property: For all $u \in \mathcal{E}'(X)$ there are an integer $k > 0$ and a number $C > 0$ such that*

$$|\mathcal{F}_u(z,\zeta)| \leq C\,(1+|\zeta|)^k \quad \text{for all } (z,\zeta) \in \mathbb{R}T'_X|_\Omega.$$

Definition 16 *Define, for any $\epsilon > 0$ and $z \in \mathbb{C}^m$,*

$$u^\epsilon(z) = \int\limits_{\mathbb{R}^m} e^{-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z,\zeta)d\zeta = \int\limits_{\mathbb{R}^m}\int\limits_{X} e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2-\epsilon\langle\zeta\rangle^2}u(z')\,\triangle\,(z-z',\zeta)dz'd\zeta$$

(of course, since $\zeta \in \mathbb{R}^m$, we have $\langle\zeta\rangle = |\zeta|$). Observe that for each fixed $\epsilon > 0$, $u^\epsilon \in \mathcal{O}\left(\mathbb{C}^m\right)$.

Theorem 17 *(FBI Inversion Formula) Suppose that $X \subset \mathbb{C}^m$ is a maximally real submanifold, $0 \in X$, and X is well-positioned at the origin. There is a neighborhood Ω of 0 in X such that*

$$\text{whatever } u \in \mathcal{E}'(\Omega),\ u(z) = (2\pi)^{-m}\lim_{\epsilon\downarrow 0}u^\epsilon(z) \text{ in } \mathcal{D}'(\Omega).$$

Remark 18 *Suppose that $X \subset \mathbb{C}^m$ is a maximally real submanifold, and X is well-positioned at the origin. Thanks to the property that $|\mathrm{Im}\,\zeta| < \tau\,|\mathrm{Re}\,\zeta|$ we can, for each $z, z' \in \Omega$, deform the domain of ζ-integration in the integral at the right in Definition 16 from $\mathbb{R}^m$ to $\mathbb{R}T'_X|_{z'}$ within the cone $\mathcal{C}_\tau$. We conclude that the integration with respect to (z',ζ) in that same integral can be carried out over $\mathbb{R}T'_X$.*

Finally, we will use the following "*Paley-Wiener*" theorem in our proof of Theorem 34:

Theorem 19 *(Theorem IX.4.1 in [T]) Let $X \subset \mathbb{C}^m$ be a maximally real submanifold passing through, and well-positioned at the origin. Let $\Omega \subset X$ be a sufficiently small neighborhood of 0 and $u \in \mathcal{E}'(\Omega)$. Then the following are equivalent:*

(i) u is C^∞ in some neighborhood Ω' of 0 in Ω;

(ii) There is a compact neighborhood K of 0 in Ω such that the following is true:

$$\begin{aligned} \text{For any integer } k &\geq 0 \text{ there exists a constant } C_k > 0 \text{ such that} \\ |\mathcal{F}_u(z,\zeta)| &\leq C_k\,(1+|\zeta|)^{-k} \quad \text{for all } (z,\zeta) \in \mathbb{R}T'_X|_K. \end{aligned}$$

1.4 Wedges in $\mathbb{C}^N$ with Generic CR Edges and the Hypo-analytic Wavefront Set

Definition 20 *Let $M \subset \mathbb{C}^N$ be a C^∞ generic CR submanifold of codimension d and CR-dimension n (so that $N = n + d$) and let $p_0 \in M$. Let $\rho = (\rho_1, ..., \rho_d)$ be a defining function of M near p_0 and V a small neighborhood of p_0 in $\mathbb{C}^N$ in which ρ is defined. If $\Gamma \subset \mathbb{R}^d$ is an open convex cone with vertex at the origin, we define*

$$\mathcal{W}(V, \rho, \Gamma) = \{Z \in V : \rho\left(Z, \overline{Z}\right) \in \Gamma\}.$$

This is an open subset of $\mathbb{C}^N$ whose boundary contains $M \cap V$. Such a set is called a wedge with edge M in the direction of Γ centered at p_0.

Example 21 *If $M \subset \mathbb{C}^N$ is a hypersurface; i.e., $d = 1$, a wedge with edge M centered at p_0 is just a one-sided neighborhood of p_0; i.e., an open set of the form*

$$\left\{Z \in V : \rho\left(Z, \overline{Z}\right) > 0\right\} \text{ or } \left\{Z \in V : \rho\left(Z, \overline{Z}\right) < 0\right\}.$$

Definition 20 is, in a sense, independent of the choice of ρ :

Lemma 22 *(Proposition 7.1.2 of [BER]) Let ρ and ρ' be two defining functions of M near p_0. Then there is a $d \times d$ real invertible matrix B such that for every V and Γ as above the following holds: For every open convex cone $\Gamma_1 \subset \mathbb{R}^d$ with $B\Gamma_1 \cap \mathbb{S}^{d-1} \subset\subset \Gamma \cap \mathbb{S}^{d-1}$, there exists an open neighborhood V_1 of p_0 in $\mathbb{C}^N$ such that $\mathcal{W}(V_1, \rho', \Gamma_1) \subset \mathcal{W}(V, \rho, \Gamma)$.*

Definition 23 *We say that a holomorphic function $f(Z) \in \mathcal{O}(\mathcal{W}(V, \rho, \Gamma))$ is of tempered growth (or slow growth) if there exists a constant $C > 0$ and an integer $k \geq 0$ such that*

$$|f(Z)| \leq \frac{C}{\left|\rho(Z, \overline{Z})\right|^k} \quad \text{for all } Z \in \mathcal{W}(V, \rho, \Gamma).$$

If $dist(Z, M)$ denotes the distance from a point Z to the submanifold M, then the above inequality is equivalent to

$$|f(Z)| \leq \frac{C}{|dist(Z, M)|^k} \quad \text{for all } Z \in \mathcal{W}(V, \rho, \Gamma),$$

where $C > 0$ might be different from the one above.

Now, if $M \subset \mathbb{C}^N$ is a C^∞ generic CR submanifold of codimension d and CR-dimension n, with $0 \in M$, then, near 0 in M, we can find holomorphic coordinates

$$Z = (z, w) = (x + iy, s + it) \in \mathbb{C}^n \times \mathbb{C}^d,$$

so that near 0,

$$M = \{(z, s + i\varphi(z, \overline{z}, s))\},$$

where $\varphi(0) = 0$ and $D\varphi(0) = 0$. As a defining function of M near 0, say in $V \subset \mathbb{C}^N$, we can take

$$\rho = (\rho_1, ..., \rho_d) = (t_1 - \varphi_1(z, \overline{z}, s), ..., t_d - \varphi_d(z, \overline{z}, s)).$$

So, if we let $\Gamma \subset \mathbb{R}^d$ be an open convex cone with vertex at the origin, then

$$\begin{aligned} \mathcal{W} &= \mathcal{W}(V, \rho, \Gamma) \\ &= \{(z, s + it) : t = \varphi(z, \overline{z}, s) + v, v \in \Gamma_\epsilon, \ |z|, \ |s| < \epsilon\} \\ &= \{(z, s + i\varphi(z, \overline{z}, s) + iv) : v \in \Gamma_\epsilon, \ |z|, \ |s| < \epsilon\} \end{aligned}$$

will be a wedge with edge M in the direction of Γ centered at 0. Thus, a function $f \in \mathcal{O}(\mathcal{W})$ is of tempered growth if there exists a constant $C > 0$ and an integer $k \geq 0$ such that

$$|f(z, s + i\varphi(z, \overline{z}, s) + iv)| \leq \frac{C}{|v|^k},$$

for all sufficiently small $z \in \mathbb{C}^n$, $s \in \mathbb{R}^d$, and $v \in \Gamma$.

A holomorphic function $f \in \mathcal{O}(\mathcal{W})$ of tempered growth has a distribution boundary value on the edge M :

Theorem 24 *(Theorem 7.2.6 of [BER]) Suppose $f \in \mathcal{O}(\mathcal{W})$ is of tempered growth. Then for any $\chi = \chi(x, y, s) \in C_0^\infty(\mathbb{R}^{2n+d})$ supported in $|z| < \epsilon$, $|s| < \epsilon$, we have that*

$$\lim_{\Gamma \ni v \to 0} \int_{\mathbb{R}^{2n+d}} f(z, s + i\varphi(z, \overline{z}, s) + iv)\chi(x, y, s) dx dy ds = \langle bf, \chi \rangle \text{ exists,}$$

and $u = bf$ is a distribution of order less than or equal to $k + 1$. In addition, uniqueness holds; i.e., if $u = bf \equiv 0$, then $f \equiv 0$. The boundary value $u = bf$ is independent of the choice of regular coordinates.

In Chapter II, namely in Theorem 45, we shall prove a more general version of the above theorem. Next, we state a converse to Theorem 24:

Theorem 25 *Suppose $f \in \mathcal{O}(\mathcal{W})$ and $u = bf$ exists in $\mathcal{D}'^k(M)$. Then in a slightly smaller wedge*

$$\mathcal{W}' = \{(z, s + i\varphi(z, \overline{z}, s) + iv) : v \in \Gamma'_{\epsilon'} \subset\subset \Gamma_\epsilon, \ \ |z|, \ \ |s| < \epsilon' < \epsilon\},$$

we have

$$|f(z, s + i\varphi(z, \overline{z}, s) + iv)| \leq \frac{C}{|v|^l} \text{ in } \mathcal{W}'$$

for some constant $C > 0$ and an integer $l \geq 0$.

Definition 26 *Given a wedge $\mathcal{W}$ in $\mathbb{C}^m$ with edge M and a point $p \in M$, we define the direction wedge $\Gamma_p(\mathcal{W}) \subset T_p\mathbb{C}^m$ to be the interior of the set*

$$\{c'(0) |\ c : [0, 1) \to \mathbb{C}^m \text{ is a } C^\infty \text{ curve satisfying } c(t) \in \mathcal{W} \text{ for } t > 0 \text{ and } c(0) = p\}.$$

Note that $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathbb{C}^m$ with edge T_pM.

Example 27 *Let $N = m+d$, $M \subset \mathbb{C}^N$ be a generic CR submanifold of codimension d with $0 \in M$. Then in a neighborhood of 0 in $\mathbb{C}^N$,*

$$M = \{(z, s + i\varphi(z, \overline{z}, s)) : \ z \in U \subset \mathbb{C}^m, \ s \in V \subset \mathbb{R}^d\},$$

where $\varphi(0) = 0$ and $D\varphi(0) = 0$. Let $\Gamma \subset \mathbb{R}^d$ be an acute open convex cone and

$$\mathcal{W} = \{(z, s + i\varphi(z, \overline{z}, s) + iv) : \ z \in U \subset \mathbb{C}^m, \ s \in V \subset \mathbb{R}^d, \ v \in \Gamma\}.$$

Then $\mathcal{W}$ is a wedge in $\mathbb{C}^N$ with edge M, and

$$\Gamma_0(\mathcal{W}) = T_0M + i\Gamma \subset T_0\mathbb{C}^N.$$

In particular, if $X \subset \mathbb{C}^m$ is a maximally real submanifold with $0 \in X$, then in a neighborhood of 0 in $\mathbb{C}^m$,

$$X = \{(x + i\Phi(x)) : \ x \in U \subset \mathbb{R}^m\},$$

where $\Phi(0) = 0$ *and* $D\Phi(0) = 0$. *Let* $\Gamma \subset \mathbb{R}^m$ *be an acute open convex cone and*

$$\mathcal{W} = \{(x + i\Phi(x) + iv) : x \in U \subset \mathbb{R}^m,\ v \in \Gamma\}.$$

Then $\mathcal{W}$ *is a wedge in* $\mathbb{C}^m$ *with edge* X, *and*

$$\Gamma_0(\mathcal{W}) = T_0X + i\Gamma \subset T_0\mathbb{C}^m.$$

Definition 28 *Let* $X \subset \mathbb{C}^m$ *be a maximally real submanifold,* $u \in \mathcal{D}'(X)$, $p \in X$, *and* $\sigma \in T_p^*X\backslash 0$. *We say that* u *is microlocally hypo-analytic at* σ *if there are acute open convex cones* $\Gamma_1, ..., \Gamma_N$ *in* T_pX, *satisfying:* $\sigma(v) < 0$ *for all* $v \in \Gamma_j$ $(1 \le j \le N)$, *and wedges* $\mathcal{W}_1, ..., \mathcal{W}_N$ *in* $\mathbb{C}^m$ *with edge* X *such that* $J\Gamma_j \subset \Gamma_p(\mathcal{W}_j)$ *and for all* $1 \le j \le N$, *there are holomorphic functions* $f_j \in \mathcal{O}(\mathcal{W}_j)$, *such that* bf_j *exists and such that* $u = \sum_{j=1}^{N} bf_j$ *on* X.

Definition 29 *The hypo-analytic wave-front set* $WF^X(u)$ *of* u *is the complement in* $T^*X\backslash 0$ *of the set of points at which* u *is microlocally hypo-analytic. It is a closed conic subset of* $T^*X\backslash 0$. *We set* $WF_p^X(u) = T_p^*X \cap WF^X(u)$.

Proposition 30 *Let* $X \subset \mathbb{C}^m$ *be a maximally real submanifold passing through, and well-positioned at* 0. *Near* 0, *we may write* $X = \{(x + i\Phi(x)) :\ x \in U \subset \mathbb{R}^m\}$, *where* $\Phi(0) = 0$ *and* $D\Phi(0) = 0$ *so that* $T_0X = \mathbb{R}^m$ *and hence,* $T_0^*X \cong \mathbb{R}^m$. *Let* $u \in \mathcal{E}'(X)$ *and suppose that* $\xi_0 \notin WF_0^X(u)$. *Then there is a neighborhood* V *of* 0 *in* $\mathbb{C}^m$, *an open cone* $\mathcal{C}$ *in* $\mathbb{C}^m\backslash 0$ *containing* ξ_0, *and constants* $c_1, c_2 > 0$ *such that*

$$|\mathcal{F}_u(z,\zeta)| \le c_1 e^{-c_2|\zeta|} \quad \textit{for all } (z,\zeta) \in V \times \mathcal{C}.$$

Proof. If u vanishes identically in a neighborhood of 0 in X, then the result follows easily; so we can assume that $u \in \mathcal{E}'(\Omega)$ where $\Omega \subset X$ is an open neighborhood of 0 as small as we wish. Since $\xi_0 \notin WF_0^X(u)$, we may assume, see Remark 31 at the end of the proof, that there is an acute open convex cone Γ in $T_0X = \mathbb{R}^m$, satisfying $\xi_0 \cdot \Gamma < 0$, a wedge $\mathcal{W}$ in $\mathbb{C}^m$ with edge X such that $J\Gamma \subset \Gamma_0(\mathcal{W})$ (in this case, the wedge has the form $\mathcal{W} = \{(x + i\Phi(x) + iv) : x \in U \subset \mathbb{R}^m,\ v \in \Gamma_\delta\}$), and a holomorphic function $f \in \mathcal{O}(\mathcal{W})$ such that $u = bf$ on Ω. Fix $v_0 \in \Gamma$ and let $c > 0$ be such that

$$\frac{\xi_0}{|\xi_0|} \cdot \frac{v_0}{|v_0|} = -c < 0.$$

Let $\Omega \subset X$ be an open neighborhood of 0 so that $\Omega \subset B_{c/8}(0) \cap X$ and the requirement for being well-positioned at 0 is satisfied for some τ, $0 < \tau < 1$. As we mentioned above, we may assume that $u \in \mathcal{E}'(\Omega)$. Recall that the FBI transform of u,

$$\mathcal{F}_u(z,\xi) = \int_\Omega e^{i\xi\cdot(z-z')-|\xi|\langle z-z'\rangle^2} u(z') \triangle (z-z',\xi)dz'.$$

Since $u = bf$ on X, we can write

$$\mathcal{F}_u(z,\xi) = \lim_{\lambda\downarrow 0} \int_\Omega g(z')e^{i\xi\cdot(z-z')-|\xi|\langle z-z'\rangle^2} f(z'+i\lambda\frac{v_0}{|v_0|}) \triangle (z-z',\xi)dz',$$

where $g \in C_0^\infty(X)$ is such that $g \equiv 1$ in a neighborhood of Ω in X. Introduce $\chi \in C_0^\infty(\Omega)$ so that $0 \le \chi \le 1$ and $\chi \equiv 1$ near 0 in Ω; say $\chi \equiv 1$ on $B_{c/16}(0) \cap \Omega$. Define for some $s > 0$, to be determined later,

$$\widetilde{z} = \widetilde{z}(z') = z' + is\chi(z')\frac{v_0}{|v_0|} \quad \text{for } z' \in \Omega.$$

Make sure that s and λ are small enough so that

$$\widetilde{z} + i\lambda\frac{v_0}{|v_0|} \in \mathcal{W} \quad \text{for all } z' \in \Omega.$$

For a fixed $\lambda > 0$ which is small enough, we can use Stokes' theorem to deform contour in the z'-variable and get that

$$\mathcal{F}_u(z,\xi) = \lim_{\lambda\downarrow 0} \int_\Omega g(\widetilde{z})e^{i\xi\cdot(z-\widetilde{z})-|\xi|\langle z-\widetilde{z}\rangle^2} f(\widetilde{z}+i\lambda\frac{v_0}{|v_0|}) \triangle (z-\widetilde{z},\xi_0)d\widetilde{z}.$$

Let

$$\begin{aligned} Q(z,z',\xi) &= i\xi\cdot(z-\widetilde{z}) - |\xi|\langle z-\widetilde{z}\rangle^2 \\ &= i\xi\cdot\left(z-z'-is\chi(z')\frac{v_0}{|v_0|}\right) - |\xi|\left\langle z-z'-is\chi(z')\frac{v_0}{|v_0|}\right\rangle^2 \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Re}\{Q(0,z',\xi_0)\} &= \mathrm{Re}\left\{-i\xi_0\cdot z' - |\xi_0|\left\langle z'\right\rangle^2\right\} + |\xi_0|(-cs\chi(z') + s^2\chi(z')^2 - 2s\chi(z')z'\cdot\frac{v}{|v|}) \\ &\le -(1-\tau)|\xi_0|\left|z'\right|^2 + |\xi_0|(-cs\chi(z') + s^2\chi(z')^2 + 2s\chi(z')\left|z'\right|). \end{aligned}$$

Hence,

$$\mathrm{Re}\{Q(0,z',\frac{\xi_0}{|\xi_0|})\} \le -(1-\tau)\left|z'\right|^2 - s\chi(z')\left[c - \left(s\chi(z') + 2\left|z'\right|\right)\right].$$

We have two cases:

$$\left|z'\right| \leq \frac{c}{16} \text{ and } \frac{c}{16} < \left|z'\right| < \frac{c}{8}.$$

· If $|z'| \leq \frac{c}{16}$, then $\chi(z') = 1$ and so, for $s < \frac{3c}{8}$ we have

$$\mathrm{Re}\{Q(0, z', \frac{\xi_0}{|\xi_0|})\} \leq -s\left[c - \left(s + 2\left|z'\right|\right)\right] < 0.$$

· If $\frac{c}{16} < |z'| < \frac{c}{8}$, then, it is easily checked that for $s < \frac{3c}{8}$,

$$\mathrm{Re}\{Q(0, z', \frac{\xi_0}{|\xi_0|})\} \leq -(1-\tau)\left|z'\right|^2 < 0.$$

Therefore, if we fix $s < \frac{3c}{8}$, then we get that for all $z' \in \Omega$,

$$\mathrm{Re}\{Q(0, z', \frac{\xi_0}{|\xi_0|})\} < -c_3, \quad \text{where } c_3 > 0.$$

Thus, by continuity of $\mathrm{Re}\, Q$, we can find an open neighborhood V of 0 in $\mathbb{C}^m$, an open cone $\mathcal{C}$ in $\mathbb{C}^m \backslash 0$ containing ξ_0, such that

$$\mathrm{Re}\{Q(z, z', \zeta)\} \leq -\frac{c_3}{2}\left|\zeta\right| \text{ for all } (z, \zeta) \in V \times \mathcal{C}, \text{ and } z' \in \Omega.$$

Note that since $u = bf$, one can find a $\lambda_0 > 0$ and $C > 0$ such that for all $0 < \lambda < \lambda_0$,

$$\left|\left\langle f(\widetilde{z} + i\lambda\frac{v_0}{|v_0|}), \varphi(z')\right\rangle\right| \leq C \sum_{|\alpha| \leq \mathrm{order}(u)} \left\|D^\alpha \varphi(z')\right\| \text{ for all } \varphi \in C_0^\infty(\Omega).$$

In our present case, $\varphi(z') = g(\widetilde{z})e^{i\xi\cdot(z-\widetilde{z}) - |\xi|\langle z-\widetilde{z}\rangle^2} \triangle(z-\widetilde{z}, \xi_0)$ and hence, for all $(z, \zeta) \in V \times \mathcal{C}$, $|\mathcal{F}_u(z, \zeta)| \leq c_1 e^{-c_2|\zeta|}$. ■

Remark 31 *We proved the result for $u = bf$. So, if $u = \sum_{j=1}^{N} bf_j$, then the result holds for each bf_j and using linearity of the FBI transform, we get our result for u.* ■

There is a converse to Proposition 30:

Proposition 32 *Let $X \subset \mathbb{C}^m$ be a maximally real submanifold passing through, and well-positioned at 0 and suppose that near 0, X has the form given in the previous proposition. Let $u \in \mathcal{E}'(X)$ and suppose that there is a neighborhood V of 0 in $\mathbb{C}^m$, an open cone $\mathcal{C}$ in $\mathbb{C}^m \backslash 0$ containing ξ_0, and constants $c_1, c_2 > 0$ such that $|\mathcal{F}_u(z, \zeta)| \leq c_1 e^{-c_2|\zeta|}$ for all $(z, \zeta) \in V \times \mathcal{C}$. Then $\xi_0 \notin WF_0^X(u)$.*

Proof. We may assume, as in the proof of Proposition 30, that $u \in \mathcal{E}'(\Omega)$, where $\Omega \subset X$ is a small enough open neighborhood of 0 for which the requirement for being well-positioned is satisfied for some $0 < \tau < 1$ and for which Theorem 17 holds so that we can use the inversion formula. Shrink Ω, if necessary, so that in Ω, X is the image of some open neighborhood U of 0 in $\mathbb{R}^m$ under the map $x \to Z(x)$ with

$$Z(x) = x + i\Phi(x),$$

where $\Phi : U \to \mathbb{R}^m$ is C^∞, $\Phi(0) = 0$, and $\Phi_x(0) = 0$. Then we have

$$u(z) = (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^m} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi,$$

where

$$\mathcal{F}_u(z,\xi) = \int_{\Omega} e^{i\xi\cdot(z-z')-|\xi|\langle z-z'\rangle^2} u(z') \triangle (z-z',\xi) dz'$$

is the FBI transform of u. Let

$$\Gamma = \mathcal{C} \cap \mathbb{R}^m,$$

an acute open convex cone in $\mathbb{R}^m$. We can write

$$u(z) = u_1(z) + u_2(z),$$

where

$$\begin{aligned} u_1(z) &= (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{\Gamma} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi, \text{ and} \\ u_2(z) &= (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^m \setminus \Gamma} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi. \end{aligned}$$

By the exponential decay of the FBI transform of u in Γ, we obtain at once that $u_1(z)$ is the restriction in $\Omega \cap V$ of a holomorphic function $f \in \mathcal{O}(V)$. For $u_2(z)$, we do the following: Write

$$\mathbb{R}^m \setminus \Gamma = \bigcup_{j=1}^{N} \overline{C}_j,$$

where each C_j is an acute open convex cone, such that

(i) $C_j \cap C_l = \varnothing$ for all $j \neq l$; and

(ii) $\Gamma_j = \{v \in T_0X : \xi \cdot v > 0 \text{ for all } \xi \in C_j \text{ and } \xi_0 \cdot v < 0\}$ is a nonempty acute open convex cone.

Shrink Γ_j, if necessary, so that one can find a constant $c > 0$ such that

$$\xi \cdot v \geq c\,|\xi|\,|v| \quad \text{for all } \ (v, \xi) \in \Gamma_j \times C_j.$$

We can write

$$u_2(z) = u_{21}(z) + \cdots + u_{2N}(z),$$

where

$$u_{2j}(z) = (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int\limits_{C_j} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z, \xi) d\xi \quad \text{for } j = 1, ..., N.$$

Define, for $j = 1, ..., N$, and for $\delta > 0$ (to be determined later):

$$\mathcal{W}_j = \left\{Z(x) + iv : x \in U, v \in (\Gamma_j)_\delta\right\} = \left\{x + i\Phi(x) + iv : x \in U, v \in (\Gamma_j)_\delta\right\}.$$

Then $\mathcal{W}_j$ is a wedge in $\mathbb{C}^m$ with edge X such that

$$J\Gamma_j \subset \Gamma_0(\mathcal{W}_j).$$

For $z = Z(x) + iv \in \mathcal{W}_j$ define

$$f_j(z) = (2\pi)^{-m} \int\limits_{\Omega} \int\limits_{C_j} e^{i\xi\cdot(z - Z(y)) - |\xi|\langle z - Z(y)\rangle^2} u(Z(y)) \bigtriangleup (z - Z(y), \xi) d\xi dZ(y).$$

We claim the following: (for a proof, see Remark 36 in the next section):

(i) $f_j \in \mathcal{O}(\mathcal{W}_j)$;

(ii) There exist $C > 0$ and an integer $k \geq 0$ such that $|f_j(z)| \leq \frac{C}{|v|^k}$ for all $z = Z(x) + iv \in \mathcal{W}_j$; and

(iii) Hence, bf_j exists in $\mathcal{D}'(\Omega)$ and we claim that it equals u_{2j}.

To sum up, we have proved that there are acute open convex cones $\Gamma_1, ..., \Gamma_N$ in T_0X, satisfying: $\xi_0 \cdot v < 0$ for all $v \in \Gamma_j$ $(1 \leq j \leq N)$ and wedges $\mathcal{W}_1, ..., \mathcal{W}_N$ in $\mathbb{C}^m$ with edge X such that $J\Gamma_j \subset \Gamma_0(\mathcal{W}_j)$ and for all $1 \leq j \leq N$, there are holomorphic functions $f_j \in \mathcal{O}(\mathcal{W}_j)$, such that bf_j exists and such that $u = \sum_{j=1}^{N} bf_j$ on X. Thus, by Definition 28, $\xi_0 \notin WF_0^X(u)$. ■

1.5 Extendability

Definition 33 *If V is a vector space and $\Gamma \subset V$ is a cone, we define the polar Γ^0, a closed convex cone in $V^*\backslash 0$, by*

$$\Gamma^0 = \{\xi \in V^*\backslash 0 : \xi(v) \geq 0 \text{ for all } v \in \Gamma\}.$$

Theorem 34 *(Proposition II.5 in [EG]) Let $\Gamma_1, ..., \Gamma_N$ be acute open convex cones in T_pX and let $u \in \mathcal{D}'(X)$. The following two properties are equivalent:*

(1) $WF_p^X(u) \subset \bigcup_{j=1}^{N} \Gamma_j^0$;

(2) Given for each $j = 1, ..., N$ a nonempty acute open convex cone $\widetilde{\Gamma}_j$ in T_pX whose closure is contained in Γ_j, there are wedges $\mathcal{W}_j$ in $\mathbb{C}^m$ with edge X such that $J\widetilde{\Gamma}_j \subset \Gamma_p(\mathcal{W}_j)$, and holomorphic functions $f_j \in \mathcal{O}(\mathcal{W}_j)$, of tempered growth, such that $u = \sum_{j=1}^{N} bf_j$ on X.

Proof. *(1) $\Longrightarrow$ (2)*: Assume that $0 \in X$ and that X is well-positioned at the origin. Let $\Omega \subset X$ be an open neighborhood of 0 and let $\tau,\ 0 < \tau < 1$, be such that

$$\begin{aligned} \text{Whatever } z, z' &\in \Omega \text{ and } \zeta \in \left(\mathbb{R}T'_X|_z\right) \cup \left(\mathbb{R}T'_X|_{z'}\right); \\ |\mathrm{Im}\,\zeta| &< \tau\,|\mathrm{Re}\,\zeta|; \\ \mathrm{Im}\left[\zeta \cdot \left(z - z'\right) + i\,\langle\zeta\rangle\,\langle z - z'\rangle^2\right] &\geq (1-\tau)\,|\zeta|\,\left|z - z'\right|^2. \end{aligned}$$

Shrink Ω, if necessary, so that in Ω, X is the image of some open neighborhood U of 0 in $\mathbb{R}^m$ under the map $x \to Z(x)$ with

$$Z(x) = x + i\Phi(x);$$

where $\Phi : U \to \mathbb{R}^m$ is C^∞, $\Phi(0) = 0$, and $\Phi_x(0) = 0$. (We can achieve $|\Phi(x)| \leq const.\,|x|^{k+1}$ for any $k \geq 2$). For each $j = 1, ..., N$, let $\widetilde{\Gamma}_j$ be as in the statement of the theorem, and let C_j be an acute open convex cone in $T_0^*X\backslash 0 \cong \mathbb{R}^m\backslash 0$ such that

$$\Gamma_j^0 \subset C_j \subset \overline{C_j} \subset \left(\widetilde{\Gamma}_j^0\right)^{int} \subset \widetilde{\Gamma}_j^0.$$

Then one can find $c > 0$ such that

$$\xi \cdot v \geq c\,|\xi|\,|v| \quad \text{for all } (v,\xi) \in \widetilde{\Gamma}_j \times C_j.$$

Shrink Ω again, if necessary, so that

$$\left|z - z'\right| < \frac{1}{16}c \text{ for all } z, z' \in \Omega, \quad \text{and} \quad |\Phi_x(x)| < \frac{c}{4+c} \text{ for all } x \in U.$$

We may assume, as we must, that $u \in \mathcal{E}'(\Omega)$ and so by the FBI inversion, we have in $\mathcal{D}'(\Omega)$,

$$u(z) = (2\pi)^{-m} \lim_{\epsilon \downarrow 0} u^{\epsilon}(z),$$

where

$$u^{\epsilon}(z) = \int_{\mathbb{R}^m} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi = \int_{\mathbb{R}^m} \int_{\Omega} e^{i\xi\cdot(z-z') - |\xi|\langle z-z'\rangle^2 - \epsilon|\xi|^2} u(z') \triangle (z-z',\xi) dz' d\xi.$$

One can write

$$u(z) = w(z) + \sum_{j=1}^{N} u_j(z),$$

where

$$\begin{aligned} w(z) &= (2\pi)^{-m} \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^m \setminus \cup_{j=1}^N C_j} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi, \text{ and} \\ u_j(z) &= (2\pi)^{-m} \lim_{\epsilon\downarrow 0} \int_{C_j \setminus \cup_{k=1}^{j-1} C_k} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi \quad \text{for } j = 1, ..., N. \end{aligned}$$

We claim that: (See Remarks 35 and 36, respectively, for proofs)

(i) w is the restriction in $\Omega \cap V$ of a holomorphic function in a neighborhood V of 0 in $\mathbb{C}^m$;

(ii) For each $j = 1, ..., N$, there is a wedge $\mathcal{W}_j$ in $\mathbb{C}^m$ with edge X such that

$$J\widetilde{\Gamma}_j \subset \Gamma_0(\mathcal{W}_j),$$

and holomorphic functions $f_j \in \mathcal{O}(\mathcal{W}_j)$, such that

$$u_j = bf_j \quad \text{in } \mathcal{D}'(\Omega).$$

Hence, the proof of the first implication is complete.

(2) $\Longrightarrow$ *(1)*: Let $\xi \in \left(T_p^*X\backslash 0\right) \setminus \bigcup_{j=1}^{N} \Gamma_j^0$. Then $\xi \notin \Gamma_j^0$ for each $j = 1, ..., N$ and so one can find $v_j \in \Gamma_j$ so that

$$\xi \cdot v_j < 0$$

and hence, one can find acute open convex cones $\widetilde{\Gamma}_j$ with $v_j \in \widetilde{\Gamma}_j \subset\subset \Gamma_j$ so that

$$\xi \cdot \widetilde{\Gamma}_j < 0 \text{ for each } j = 1, ..., N.$$

By our assumption in (2), there are wedges $\mathcal{W}_j$ in $\mathbb{C}^m$ with edge X such that $J\widetilde{\Gamma}_j \subset \Gamma_p(\mathcal{W}_j)$, and holomorphic functions $f_j \in \mathcal{O}(\mathcal{W}_j)$, of tempered growth, such that $u = \sum_{j=1}^{N} bf_j$ on X. Using Definition 28, we get that $\xi \notin WF_p^X(u)$ and so $WF_p^X(u) \subset \bigcup_{j=1}^{N} \Gamma_j^0$. ■

Remark 35 *Since* $WF_0^X(u) \subset \bigcup_{j=1}^{N} \Gamma_j^0$ *and since* $\left(\mathbb{R}^m \setminus \bigcup_{j=1}^{N} C_j\right) \cap \mathbb{S}^{m-1}$ *is compact, we can use Proposition 30 to get a neighborhood* V *of* 0 *in* $\mathbb{C}^m$*, a conic neighborhood* $\mathcal{C}$ *of* $\mathbb{R}^m \setminus \bigcup_{j=1}^{N} C_j$ *in* $\mathbb{C}^m \backslash 0$*, and constants* $c_1, c_2 > 0$ *such that*

$$|\mathcal{F}_u(z,\zeta)| \leq c_1 e^{-c_2|\zeta|} \quad \textit{for all } (z,\zeta) \in V \times \mathcal{C}.$$

For $z \in V$*, define*

$$h(z) = (2\pi)^{-m} \int_{\mathbb{R}^m \setminus \cup_{j=1}^{N} C_j} \mathcal{F}_u(z,\xi) d\xi.$$

Since $\mathcal{F}_u(z,\xi)$ *is an entire holomorphic function of* z *for each fixed* ξ*, and by the above inequality, we get that* $h \in \mathcal{O}(V)$ *and one can pass the limit under the integral sign in the expression for* w *for* $z \in \Omega \cap V$*; i.e.,*

$$w(z) = (2\pi)^{-m} \int_{\mathbb{R}^m \setminus \cup_{j=1}^{N} C_j} \mathcal{F}_u(z,\xi) d\xi \quad \textit{for } z = Z(x) \in \Omega \cap V.$$

Thus, w *is the restriction in* $\Omega \cap V$ *of a holomorphic function in a neighborhood* V *of* 0 *in* $\mathbb{C}^m$*.*

Remark 36 *For* $j = 1, ..., N$*, and for* $\delta > 0$ *(to be determined later) define*

$$\mathcal{W}_j = \left\{ Z(x) + iv : x \in U, v \in \left(\widetilde{\Gamma}_j\right)_\delta \right\} = \left\{ x + i\Phi(x) + iv : x \in U, v \in \left(\widetilde{\Gamma}_j\right)_\delta \right\}.$$

Then $\mathcal{W}_j$ is a wedge in $\mathbb{C}^m$ with edge X such that

$$J\widetilde{\Gamma}_j \subset \Gamma_0(\mathcal{W}_j).$$

For $z = Z(x) + iv \in \mathcal{W}_j$, $\xi \in C_j$, and $y \in U$, if $\zeta = \zeta(\xi) = {}^t Z_y(y)^{-1}\xi \in \mathbb{R}T'_X|_{Z(y)}$, define

$$f_j(z) = (2\pi)^{-m} \int\limits_{\Omega} \int\limits_{C_j \setminus \cup_{k=1}^{j-1} C_k} e^{i\zeta\cdot(z-Z(y)) - \langle\zeta\rangle\langle z-Z(y)\rangle^2} u(Z(y)) \triangle (z - Z(y), \zeta) d\zeta dZ(y).$$

We claim the following:

(i) $f_j \in \mathcal{O}(\mathcal{W}_j)$;

(ii) f_j is of tempered growth in $\mathcal{W}_j$; i.e., there exist $C > 0$ and an integer $k \geq 0$ such that

$$|f_j(z)| \leq \frac{C}{|v|^k} \quad \text{for all } z = Z(x) + iv \in \mathcal{W}_j; \text{ and}$$

(iii) Hence, bf_j exists in $\mathcal{D}'(\Omega)$ and we claim that it equals u_j.

Proof of claim (i): Define, for $x, y \in U$, $\xi \in C_j$, and $v \in \left(\widetilde{\Gamma}_j\right)_\delta$,

$$\begin{aligned} Q(x,y,\xi,v) &= i\zeta\cdot(Z(x)+iv-Z(y)) - \langle\zeta\rangle \langle Z(x)+iv-Z(y)\rangle^2 \\ &= \left[i\zeta\cdot(Z(x)-Z(y)) - \langle\zeta\rangle \langle Z(x)-Z(y)\rangle^2\right] - \zeta\cdot v \\ &\quad - \langle\zeta\rangle \left[2iv\cdot(Z(x)-Z(y)) - |v|^2\right]. \end{aligned}$$

Since X is well-positioned at the origin, we have $\forall x, y \in U, \xi \in C_j$

$$\operatorname{Re}\left\{i\zeta\cdot(Z(x)-Z(y)) - \langle\zeta\rangle \langle Z(x)-Z(y)\rangle^2\right\} \leq -(1-\tau)|\zeta|\,|Z(x)-Z(y)|^2.$$

Also, for all $(v,\xi) \in \widetilde{\Gamma}_j \times C_j$,

$$\begin{aligned} -\zeta\cdot v &= -\left({}^t Z_y(y)^{-1}\xi\right)\cdot v \\ &= -\xi\cdot\left(Z_y(y)^{-1}v\right) \\ &= -\xi\cdot\left([I+i\Phi_y(y)]^{-1}v\right) \\ &= -\xi\cdot\left[\sum_{k=0}^{\infty}(-i)^k(\Phi_y(y))^k v\right] \\ &= -\xi\cdot\left[v + \sum_{k=1}^{\infty}(-i)^k(\Phi_y(y))^k v\right] \\ &= -\xi\cdot v - \xi\cdot\left[\sum_{k=1}^{\infty}(-i)^k(\Phi_y(y))^k v\right]. \end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Re}\{-\zeta\cdot v\} &= -\xi\cdot v-\operatorname{Re}\left\{\xi\cdot\left[\sum_{k=1}^{\infty}(-i)^k\left(\Phi_y(y)\right)^k v\right]\right\}\\ &\le -c\,|\xi|\,|v|+\left|\xi\cdot\left[\sum_{k=1}^{\infty}(-i)^k\left(\Phi_y(y)\right)^k v\right]\right|\\ &\le -c\,|\xi|\,|v|+|\xi|\,|v|\left[\sum_{k=1}^{\infty}|\Phi_y(y)|^k\right]\\ &\le -c\,|\xi|\,|v|+|\xi|\,|v|\sum_{k=1}^{\infty}\left(\frac{c}{4+c}\right)^k\\ &= -\frac{3}{4}c\,|\xi|\,|v|.\end{aligned}$$

Thus, we have

$$\operatorname{Re}\{-\zeta\cdot v\}\le-\frac{3}{4}c\,|\xi|\,|v|\quad\forall\xi\in C_j\ \textit{and}\ v\in\left(\widetilde{\Gamma}_j\right)_\delta.$$

Finally, since $|\langle\zeta\rangle|\le|\zeta|$, *and after shrinking* Ω *further so that* $|\zeta|\le 2|\xi|$, *we have for* $\delta<\frac{1}{8}c$:

$$\begin{aligned}\operatorname{Re}\left\{-\langle\zeta\rangle\left[2iv\cdot(Z(x)-Z(y))-|v|^2\right]\right\} &\le |\langle\zeta\rangle|\left[2\,|v|\,|Z(x)-Z(y)|+|v|^2\right]\\ &\le |\zeta|\,|v|\,[2\,|Z(x)-Z(y)|+|v|]\\ &< 2\,|\xi|\,|v|\left[2\frac{c}{16}+\frac{1}{8}c\right]\\ &= \frac{1}{2}c\,|\xi|\,|v|,\quad\textit{for all}\ x,y\in U,\xi\in C_j\ \textit{and}\ v\in\left(\widetilde{\Gamma}_j\right)_\delta.\end{aligned}$$

Hence,

$$\operatorname{Re}\left\{-\langle\zeta\rangle\left[2iv\cdot(Z(x)-Z(y))-|v|^2\right]\right\}\le\frac{1}{2}c\,|\xi|\,|v|\quad\forall x,y\in U,\ \xi\in C_j\ \textit{and}\ v\in\left(\widetilde{\Gamma}_j\right)_\delta.$$

Hence, combining the above inequalities, we obtain

$$\operatorname{Re}\{Q(x,y,\xi,v)\}\le-\frac{1}{4}c\,|\xi|\,|v|\quad\forall x,y\in U,\ \xi\in C_j\ \textit{and}\ v\in\left(\widetilde{\Gamma}_j\right)_\delta.$$

Since holomorphy is a local property, we can use the last inequality, after fixing a point $z\in\mathcal{W}_j$, *to show that near the fixed point* z, *one has*

$$\operatorname{Re}\{Q(x,y,\xi,v)\}\le-c_2\,|\xi|,$$

for all v *in an appropriately chosen open set in its domain and for all* $x,y\in U,\ \xi\in C_j$. *Thus,*

$$\left|e^{i\zeta\cdot(Z(x)+iv-Z(y))-\langle\zeta\rangle\langle Z(x)+iv-Z(y)\rangle^2}u(Z(y))\bigtriangleup(Z(x)+iv-Z(y),\zeta)\right|\le c_1e^{-c_2|\xi|}\in L^1(\mathbb{R}^m)$$

and consequently, f_j is holomorphic near our fixed point in $\mathcal{W}_j$ and by randomness of our choice, we conclude that $f_j \in \mathcal{O}(\mathcal{W}_j)$.

Proof of claim (ii): We have, as we did in the proof of claim (i):

$$\begin{aligned} |f_j(Z(x)+iv)| &\le (2\pi)^{-m} \int\limits_{C_j \setminus \cup_{k=1}^{j-1} C_k} \int\limits_{\Omega} c_1 e^{-c_2|\xi||v|} (1+|\xi|)^l \, dZ(y) d\xi \\ &= c_1' \int\limits_{C_j \setminus \cup_{k=1}^{j-1} C_k} e^{-c_2|\xi||v|} (1+|\xi|)^l \, d\xi. \end{aligned}$$

From this, one can easily show that there are $C > 0$ and an integer $k \ge 0$ such that

$$|f_j(Z(x)+iv)| \le \frac{C}{|v|^k} \quad \text{for all } Z(x)+iv \in \mathcal{W}_j.$$

Proof of claim (iii): We will use the following Lemma (compare to Theorem 19):

Lemma 37 *Let $\varphi \in C_0^\infty(\Omega)$. Then for any integer $l \ge 0$ there exists a constant $d_l > 0$ such that the following holds: For any $x \in U$, if $z = Z(x)+iv \in \mathcal{W}_j$ and if $\zeta \in {}^t Z_x(x)^{-1} C_j = \{ {}^t Z_x(x)^{-1}\xi : \xi \in C_j \}$, then*

$$|\mathcal{F}_\varphi(z,\zeta)| \le d_l (1+|\zeta|)^{-l}.$$

Proof. Integration by parts gives

$$\left(1+\langle\zeta\rangle^2\right)^l \mathcal{F}_\varphi(z,\zeta) = \int\limits_X e^{i\zeta\cdot(z-z')} \left(1+\triangle_M'\right)^l \left\{ e^{-\langle\zeta\rangle\langle z-z'\rangle^2} \varphi(z') \triangle (z-z',\zeta) \right\} dz',$$

where $\triangle_M' = M_1'^2 + \cdots + M_m'^2$ and M_i' is the vector field on X denoted by M_i in Proposition I.2.1, but now acting in the variables z'. There is a constant $a_l > 0$ such that

$$\begin{aligned} &\left| e^{\langle\zeta\rangle\langle z-z'\rangle^2} \left(1+\triangle_M'\right)^l \left\{ e^{-\langle\zeta\rangle\langle z-z'\rangle^2} \varphi(z') \triangle (z-z',\zeta) \right\} \right| \\ \le \; & a_l (1+|\zeta|)^l (1+|\zeta| \left|z-z'\right|^2)^l \sum_{|\alpha|\le 2l} \left| M'^\alpha \varphi(z') \right|. \end{aligned}$$

Shrinking Ω further, if necessary, assuming that $\delta < 1/2$, and using the estimates that we had in the proof of claim (i) above, we get that for a suitable $b_l > 0$

$$\left| \left(1+\langle\zeta\rangle^2\right)^l \mathcal{F}_\varphi(z,\zeta) \right| \le b_l (1+|\zeta|)^l \int\limits_{\operatorname{supp}\varphi} \left(1+|\zeta| \left[\left|Z(x)-z'\right|^2 + |v| \right] \right)^l e^{-a|\zeta|\left[|Z(x)-z'|^2+|v|\right]} dz'.$$

The integrand in the last inequality is bounded and so, for some $c_l > 0$ we have

$$\left|\left(1+\langle\zeta\rangle^2\right)^l \mathcal{F}_\varphi(z,\zeta)\right| \le c_l(1+|\zeta|)^l.$$

Now, using the fact that $|\mathrm{Im}\,\zeta| < \tau\,|\mathrm{Re}\,\zeta|$, one can find $c_l' > 0$ such that

$$\left|1+\langle\zeta\rangle^2\right|^l \ge c_l'\,(1+|\zeta|)^{2l},$$

and consequently there is a constant $d_l > 0$ such that $|\mathcal{F}_\varphi(Z(x)+iv,\zeta)| \le d_l(1+|\zeta|)^{-l}$. ■

Now, define

$$\begin{aligned} Q &= Q(x,y,\zeta,v) = i\zeta\cdot(Z(x)+iv-Z(y)) - \langle\zeta\rangle\,\langle Z(x)+iv-Z(y)\rangle^2, \\ \triangle &= \triangle(Z(x)+iv-Z(y),\zeta) \end{aligned}$$

and let $\varphi \in C_0^\infty(\Omega)$. Then we have

$$\begin{aligned} (2\pi)^m \langle bf_j,\varphi\rangle &= (2\pi)^m \lim_{\widetilde{\Gamma}_j\ni v\to 0} \int\limits_{\mathrm{supp}\varphi} f_j(Z(x)+iv)\varphi(Z(x))dZ(x) \\ &= \lim_{\widetilde{\Gamma}_j\ni v\to 0} \int\limits_{\mathrm{supp}\varphi}\int\limits_{\Omega}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} e^Q u(Z(y))\varphi(Z(x))\triangle\, d\zeta dZ(y)dZ(x) \\ &= \lim_{\widetilde{\Gamma}_j\ni v\to 0} \int\limits_{\Omega}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} [\int\limits_{\mathrm{supp}\varphi} e^Q\varphi(Z(x))\triangle\, dZ(x)]u(Z(y))d\zeta dZ(y) \\ &= \lim_{\widetilde{\Gamma}_j\ni v\to 0} \int\limits_{\Omega}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} [\mathcal{F}_\varphi(Z(y)-iv,-\zeta)]\,u(Z(y))d\zeta dZ(y) \\ &= \int\limits_{\Omega}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} \mathcal{F}_\varphi(Z(y),-\zeta)u(Z(y))d\zeta dZ(y) \qquad \text{(by Lemma 37).} \end{aligned}$$

Now, recall that

$$u_j(z) = (2\pi)^{-m}\lim_{\epsilon\downarrow 0}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} e^{-\epsilon|\xi|^2}\mathcal{F}_u(z,\xi)d\xi,$$

and by deforming contour in the ξ-variable as we did in claim (i), we obtain

$$u_j(z) = (2\pi)^{-m}\lim_{\epsilon\downarrow 0}\int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k}\int\limits_{\Omega} e^{i\zeta\cdot(z-z')-\langle\zeta\rangle\langle z-z'\rangle^2-\epsilon\langle\zeta\rangle^2}u(z')\triangle(z-z',\zeta)dz'd\zeta.$$

Therefore, (here, $Q = i\zeta \cdot (z - z') - \langle\zeta\rangle \langle z - z'\rangle^2$ and $\triangle = \triangle(z - z', \zeta)$)

$$\begin{aligned}(2\pi)^m \langle u_j, \varphi\rangle &= \lim_{\epsilon\downarrow 0} \int\limits_{\mathrm{supp}\varphi} \int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} \int\limits_{\Omega} e^{Q-\epsilon\langle\zeta\rangle^2} u(z')\varphi(z) \triangle\, dz' d\zeta dz \\ &= \lim_{\epsilon\downarrow 0} \int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} \int\limits_{\Omega} \left[\int\limits_{\mathrm{supp}\varphi} e^{Q}\varphi(z) \triangle\, dz \right] e^{-\epsilon\langle\zeta\rangle^2} u(z') dz' d\zeta \\ &= \lim_{\epsilon\downarrow 0} \int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} \int\limits_{\Omega} \left[\mathcal{F}_{\varphi}(z', -\zeta)\right] e^{-\epsilon\langle\zeta\rangle^2} u(z') dz' d\zeta \\ &= \int\limits_{C_j\setminus\cup_{k=1}^{j-1}C_k} \int\limits_{\Omega} \mathcal{F}_{\varphi}(z', -\zeta) u(z') dz' d\zeta \qquad \text{(by Theorem 19).}\end{aligned}$$

Hence, $bf_j = u_j$ in $\mathcal{D}'(\Omega)$.

We have some corollaries to Theorem 34. The first one is just a restatement for the special case $N = 1$.

Corollary 38 *Let Γ be an acute open convex cone in T_pX and let $u \in \mathcal{D}'(X)$. The following two properties are equivalent:*

(1) $WF_p^X(u) \subset \Gamma^0$;

(2) Given a nonempty acute open convex cone $\widetilde{\Gamma}$ in T_pX whose closure is contained in Γ, there is a wedges $\mathcal{W}$ in $\mathbb{C}^m$ with edge X such that $J\widetilde{\Gamma} \subset \Gamma_p(\mathcal{W})$, and a holomorphic function $f \in \mathcal{O}(\mathcal{W})$, of tempered growth, such that $u = bf$ on X.

The second corollary to Theorem 34 is the so called Edge-of-the-Wedge Theorem:

Corollary 39 *(Edge-of-the-Wedge Theorem) Let $X \subset \mathbb{C}^m$ be a maximally real submanifold, let $p \in X$, and let $\mathcal{W}^+$ and $\mathcal{W}^-$ be wedges in $\mathbb{C}^m$ with edge X whose directions are opposite: $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$. If $u \in \mathcal{D}'(X)$ is the boundary value of a holomorphic function $f^+ \in \mathcal{O}(\mathcal{W}^+)$ and also the boundary value of a holomorphic function $f^- \in \mathcal{O}(\mathcal{W}^-)$, then $WF_p^X(u) = \varnothing$.*

Proof. Let $\Gamma \subset T_pX$ be an acute open convex cone such that $J\Gamma \subset \Gamma_p(\mathcal{W}^+)$. Then $J(-\Gamma) = -J\Gamma \subset -\Gamma_p(\mathcal{W}^+) = \Gamma_p(\mathcal{W}^-)$ and so, by Corollary 38, we get that $WF_p^X(u) \subset \Gamma^0 \cap (-\Gamma)^0$. Note that if $\xi \in \Gamma^0 \cap (-\Gamma)^0$, then $\xi \cdot \Gamma \geq 0$ and $\xi \cdot (-\Gamma) = -\xi \cdot \Gamma \geq 0$ which implies that $\xi = 0$. But recall that $WF_p^X(u) \subset T_p^*X \backslash 0$. ■

Remark 40 *The conclusion $WF_p^X(u) = \varnothing$ in Corollary 39 means that u is actually the restriction, to X, of a holomorphic function $f \in \mathcal{O}(V)$ where V is a small open neighborhood of p in $\mathbb{C}^m$. Thus, u is hypo-analytic at p. Also, by uniqueness of boundary value, we get that $f|_{V \cap \mathcal{W}^+} = f^+$ and $f|_{V \cap \mathcal{W}^-} = f^-$.*

Before we state and prove our second theorem in this section, we have the following useful lemma that will be used in the proof of the theorem:

Lemma 41 *Let $X \subset \mathbb{C}^m$ be a maximally real submanifold passing through, and well-positioned at the origin. Suppose that near the origin, X is of the form given in Proposition 30. Let $\Gamma_1, ..., \Gamma_N$ be acute open convex cones in $T_0X \backslash \{0\} = \mathbb{R}^m \backslash \{0\}$ and $u_1, ..., u_N \in \mathcal{D}'(X)$ be such that*

$$WF_0^X(u_j) \subset \Gamma_j^0 \ \ \textit{for } j = 1, ..., N.$$

Set $u = \sum_{j=1}^{N} u_j$. Then

$$WF_0^X(u) \subset \bigcup_{j=1}^{N} \Gamma_j^0.$$

Proof. Suppose that $\xi_0 \notin \bigcup_{j=1}^{N} \Gamma_j^0$. Then $\xi_0 \notin WF_0^X(u_j)$ for all $j = 1, ..., N$. Hence, by Proposition 30, for each $j = 1, ..., N$, there is a neighborhood V_j of 0 in $\mathbb{C}^m$, an open cone $\mathcal{C}_j$ in $\mathbb{C}^m \backslash 0$ containing ξ_0, and constants $c_{1j}, c_{2j} > 0$ such that

$$\left|\mathcal{F}_{u_j}(z,\zeta)\right| \leq c_{1j} e^{-c_{2j}|\zeta|} \ \text{ for all } \ (z,\zeta) \in V_j \times \mathcal{C}_j.$$

If we set

$$c_1 = \max_{1 \leq j \leq N}\{c_{1j}\}; \ c_2 = \min_{1 \leq j \leq N}\{c_{2j}\}; \ V = \bigcap_{j=1}^{N} V_j; \ \text{and } \mathcal{C} = \bigcap_{j=1}^{N} \mathcal{C}_j,$$

then we get that there is a neighborhood V of 0 in $\mathbb{C}^m$, an open cone $\mathcal{C}$ in $\mathbb{C}^m\backslash 0$ containing ξ_0, and constants $c_1, c_2 > 0$ such that for all $j = 1, ..., N$,

$$\left|\mathcal{F}_{u_j}(z,\zeta)\right| \le c_1 e^{-c_2|\zeta|} \text{ for all } (z,\zeta) \in V \times \mathcal{C}.$$

Hence,

$$|\mathcal{F}_u(z,\zeta)| = \left|\sum_{j=1}^{N} \mathcal{F}_{u_j}(z,\zeta)\right| \le N c_1 e^{-c_2|\zeta|} \text{ for all } (z,\zeta) \in V \times \mathcal{C}.$$

This implies, using Proposition 32, that $\xi_0 \notin WF_0^X(u)$, and so $WF_0^X(u) \subset \bigcup_{j=1}^{N} \Gamma_j^0$. ∎

Theorem 42 *Let $X \subset \mathbb{C}^m$ be a maximally real submanifold passing through, and well-positioned at the origin. Suppose that near the origin, X is of the form given in Proposition 30. Let $u \in \mathcal{D}'(X)$ and suppose that $\Gamma_1, ..., \Gamma_N$ are acute open convex cones in $T_0X\backslash\{0\} = \mathbb{R}^m\backslash\{0\}$ such that $\bigcup_{j=1}^{N} \Gamma_j^0 = \mathbb{R}^m\backslash\{0\} \cong T_0^*X\backslash\{0\}$. Then*

(a) *u can be decomposed as $u = \sum_{j=1}^{N} u_j$, where $u_j \in \mathcal{D}'(X)$ and*

$$WF_0^X(u_j) \subset WF_0^X(u) \cap \Gamma_j^0.$$

(b) *If $u = \sum_{j=1}^{N} u'_j$ is another such decomposition, then $u'_j = u_j + \sum_{j\neq l} u_{jl}$, with $u_{jl} \in \mathcal{D}'(X)$, $u_{jl} + u_{lj}$ is hypo-analytic, and*

$$WF_0^X(u_{jl}) \subset \Gamma_j^0 \cap \Gamma_l^0.$$

(In fact, the u_{jl}'s can be chosen so that $u_{jl} = -u_{lj}$).

Proof. (a) We may assume that

$$\left(\Gamma_j^0 \cap \Gamma_l^0\right)^{int} = \varnothing$$

(Otherwise, replace Γ_j^0 by Γ_j^{0*}, where $\Gamma_j^{0*} = \overline{\Gamma_j^0 \backslash \left(\Gamma_1^0 \cup \cdots \cup \Gamma_{j-1}^0\right)} \subset \Gamma_j^0$). For $j = 1, ..., N$, let

$$\{\Gamma_{jk} : k = 1, 2, 3, ...\}$$

be a sequence of acute open convex cones such that

$$\begin{aligned}\Gamma_{j1} &\subset \Gamma_{j2} \subset \Gamma_{j3} \subset \cdots, \\ \overline{\Gamma}_{jk} &\subset \Gamma_j \text{ for each } k = 1, 2, ..., \text{ and} \\ \bigcup_{k=1}^{\infty} \Gamma_{jk} &= \Gamma_j.\end{aligned}$$

Then, for each $k = 1, 2, 3, ...$, $\Gamma_j^0 \subset\subset \Gamma_{jk}^0$ and one can find $c = c(k) > 0$ such that

$$\xi \cdot v \geq c\,|\xi|\,|v| \quad \text{for all } (\xi, v) \in \Gamma_j^0 \times \Gamma_{jk}.$$

For $k = 1, 2, 3, ...,$ define

$$\begin{aligned}\mathcal{W}_{jk} &= \left\{Z(x) + iv : x \in U, v \in (\Gamma_{jk})_\delta\right\} \\ &= \left\{x + i\Phi(x) + iv : x \in U, v \in (\Gamma_{jk})_\delta\right\}.\end{aligned}$$

For $z = Z(x) + iv \in \mathcal{W}_{jk}$, $\xi \in \Gamma_j^0$, and $y \in U$, if $\zeta = \zeta(\xi) = {}^tZ_y(y)^{-1}\xi \in \mathbb{R}T'_X|_{Z(y)}$, define for $k = 1, 2, 3, ...$,

$$f_{jk}(z) = (2\pi)^{-m} \int\limits_{\Omega} \int\limits_{\Gamma_j^0} e^{i\zeta\cdot(z-Z(y)) - \langle\zeta\rangle\langle z-Z(y)\rangle^2} u(Z(y)) \bigtriangleup (z - Z(y), \zeta) d\zeta dZ(y)$$

As we did in Remark 36, we get that, for each $k = 1, 2, 3, ...,$ $f_{jk} \in \mathcal{O}(\mathcal{W}_{jk})$ and $bf_{jk} = u_j$, where

$$u_j(z) = u_j(Z(x)) = (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int\limits_{\Gamma_j^0} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z, \xi) d\xi.$$

Of course, $u_j \in \mathcal{D}'(X)$, and $u = \sum_{j=1}^{N} u_j$. We claim that

$$WF_0^X(u_j) \subset \Gamma_j^0.$$

To show this, suppose that $\xi_0 \notin \Gamma_j^0$. Since Γ_j^0 is closed, one can find an acute open convex cone $\Gamma'_j \subset\subset \Gamma_j$ so that

$$\xi_0 \cdot \Gamma'_j < 0.$$

Choose $k \in \mathbb{N}$ large enough so that

$$\Gamma'_j \subset\subset \Gamma_{jk}.$$

Set

$$\mathcal{W}'_j = \left\{x + i\Phi(x) + iv : x \in U, v \in \left(\Gamma'_j\right)_\delta\right\} \subset \mathcal{W}_{jk}.$$

To summerize, we have found an acute open convex cone Γ_j' in $\mathbb{R}^m\backslash 0$ satisfying

$$\xi_0 \cdot \Gamma_j' < 0,$$

and a wedge $\mathcal{W}_j'$ in $\mathbb{C}^m$ with edge X such that $J\Gamma_j' \subset \Gamma_0(\mathcal{W}_j')$ and a holomorphic function $f_{jk} \in \mathcal{O}(\mathcal{W}_j')$ such that

$$bf_{jk} = u_j.$$

Hence, using the definition of $WF_0^X(u_j)$, we get that $\xi_0 \notin WF_0^X(u_j)$ and so

$$WF_0^X(u_j) \subset \Gamma_j^0.$$

It remains to show that $WF_0^X(u_j) \subset WF_0^X(u)$. To do so, suppose that $\xi_0 \notin WF_0^X(u)$. Then, by Proposition 30, there is a neighborhood V of 0 in $\mathbb{C}^m$, an open cone $\mathcal{C}$ in $\mathbb{C}^m\backslash 0$ containing ξ_0, and constants $c_1, c_2 > 0$ such that

$$|\mathcal{F}_u(z,\zeta)| \leq c_1 e^{-c_2|\zeta|} \text{ for all } (z,\zeta) \in V \times \mathcal{C}.$$

Write

$$u_j(z) = u_{j1}(z) + u_{j2}(z),$$

where

$$\begin{aligned} u_{j1}(z) &= (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{\Gamma_j^0 \backslash \mathcal{C}} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi; \text{ and} \\ u_{j2}(z) &= (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{\Gamma_j^0 \cap \mathcal{C}} e^{-\epsilon|\xi|^2} \mathcal{F}_u(z,\xi) d\xi. \end{aligned}$$

Thanks to the exponential decay of the FBI transform of u in $V \times \mathcal{C}$, we get that u_{j2} is the restriction of a holomorphic function in a small neighborhood of 0 in $\mathbb{C}^m$ and so $WF_0^X(u_{j2}) = \varnothing$. Hence,

$$WF_0^X(u_j) \subset WF_0^X(u_{j1}).$$

Using the same argument which showed that $WF_0^X(u_j) \subset \Gamma_j^0$, we get that

$$WF_0^X(u_{j1}) \subset \Gamma_j^0 \backslash \mathcal{C},$$

and so

$$\xi_0 \notin WF_0^X(u_{j1}).$$

Therefore, $\xi_0 \notin WF_0^X(u_j)$ and we conclude that $WF_0^X(u_j) \subset WF_0^X(u) \cap \Gamma_j^0$.

(b) We claim that (see Remark 43 for a proof)

$$WF_0^X(u_j' - u_j) \subset \bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0\right).$$

We may assume that $\Gamma_j^0 \cap \Gamma_l^0 \cap \Gamma_k^0 = \varnothing$ whenever $1 \leq j < l < k \leq N$. (Otherwise, replace Γ_k^0 with $\Gamma_k^{0*} = \Gamma_k^0 \backslash \Gamma_k' \subset \Gamma_k^0$ where Γ_k' is an acute open convex cone which contains $\Gamma_j^0 \cap \Gamma_l^0$). Then, one can find acute open convex cones $C_{jl},\ j \neq l$, whose closures are distinct, such that

$$\Gamma_j^0 \cap \Gamma_l^0 \subset C_{jl}.$$

Hence, by our claim,

$$WF_0^X(u_j' - u_j) \subset \bigcup_{l \neq j} C_{jl}.$$

Write

$$\mathbb{R}^m \backslash \{0\} = \left(\bigcup_{l \neq j = 1}^{N} \overline{C_{jl}} \right) \cup \left(\bigcup_{j=1}^{N'} W_j \right),$$

where each W_j is a closed convex cone. (This can be done by writing $(\mathbb{R}^m \backslash \{0\}) \setminus \left(\bigcup_{l \neq j=1}^{N} \overline{C_{jl}} \right)$ as a union of acute open convex cones and then taking closures of these cones). Now, we claim that if $\mathcal{C} \subset \mathbb{R}^m \backslash \{0\}$ is a closed convex cone, then

$$\mathcal{C} = \left(\left(\mathcal{C}^0\right)^{int} \right)^0.$$

To show this, let $v \in \mathcal{C}$. Then $v \cdot \mathcal{C}^0 \geq 0$ and in particular, $v \cdot \left(\mathcal{C}^0\right)^{int} \geq 0$. Hence, $v \in \left(\left(\mathcal{C}^0\right)^{int} \right)^0$. On the other hand, if $v \notin \mathcal{C}$, then one can find an acute open convex cone $\mathcal{C}' \subset\subset \mathcal{C}^0$ such that $v \cdot \mathcal{C}' < 0$. Hence, $v \notin \left(\left(\mathcal{C}^0\right)^{int} \right)^0$ and the claim follows. This allows us, using part (a), to write

$$u_j' - u_j = \sum_{l \neq j = 1}^{N} u_{jl} + \sum_{j=1}^{N'} v_j,$$

where (note here that $\bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0\right) \cap W_j \subset \left(\bigcup_{l \neq j} C_{jl} \right) \cap W_j = \varnothing$)

$$\begin{aligned} WF_0^X(u_{jl}) &\subset WF_0^X(u_j' - u_j) \cap \overline{C_{jl}} \subset \bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0\right) \cap \overline{C_{jl}} = \Gamma_j^0 \cap \Gamma_l^0; \text{ and} \\ WF_0^X(v_j) &\subset WF_0^X(u_j' - u_j) \cap W_j \subset \bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0\right) \cap W_j = \varnothing. \end{aligned}$$

If one ignores the v_j's (since they are hypo-analytic after all) by adding them to one of the u_{jl}'s, then one gets that

$$u_j' - u_j = \sum_{l \neq j=1}^{N} u_{jl},$$

with $u_{jl} \in \mathcal{D}'(X)$ and

$$WF_0^X(u_{jl}) \subset \Gamma_j^0 \cap \Gamma_l^0.$$

Now, it remains to show that $u_{jl} + u_{lj}$ is hypo-analytic, or in other words,

$$WF_0^X(u_{jl} + u_{lj}) = \varnothing.$$

To do so, fix j and l, $j \neq l$, $1 \leq j, l \leq N$. For ease of notation let $\{p \neq q\}^*$ denote the statement:

$$\{p, q\} \cap \{j, l\} \neq \{j, l\} \quad \text{and} \quad 1 \leq p \neq q \leq N.$$

Note that

$$\sum_{j=1}^{N} \left(u_j' - u_j\right) = 0 \Rightarrow \sum_{j=1}^{N} \sum_{l \neq j} u_{jl} = 0 \Rightarrow \sum_{l \neq j} (u_{jl} + u_{lj}) = 0 \Rightarrow u_{jl} + u_{lj} = - \sum_{\{p \neq q\}^*} (u_{pq} + u_{qp}).$$

But by Lemma 41,

$$\begin{aligned} WF_0^X(u_{jl} + u_{lj}) &\subset \Gamma_j^0 \cap \Gamma_l^0; \text{ and} \\ WF_0^X\left(- \sum_{\{p \neq q\}^*} (u_{pq} + u_{qp})\right) &\subset \bigcup_{\{p \neq q\}^*} \left(\Gamma_p^0 \cap \Gamma_q^0\right). \end{aligned}$$

Hence,

$$WF_0^X(u_{jl} + u_{lj}) \subset \left(\Gamma_j^0 \cap \Gamma_l^0\right) \cap \left(\bigcup_{\{p \neq q\}^*} \left(\Gamma_p^0 \cap \Gamma_q^0\right)\right) = \varnothing.$$

Therefore, $u_{jl} + u_{lj}$ is hypo-analytic in X. ■

Remark 43 *By Lemma 41, we know that*

$$WF_0^X(u_j' - u_j) \subset \Gamma_j^0.$$

So, it suffices to prove that

$$WF_0^X(u_j' - u_j) \cap \left[\Gamma_j^0 \backslash \bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0\right)\right] = \varnothing.$$

To do so, fix $j \in \{1, ..., N\}$ *and suppose that* $\xi_0 \in \Gamma_j^0 \backslash \bigcup_{l \neq j} \left(\Gamma_j^0 \cap \Gamma_l^0 \right)$. *Then* $\xi_0 \notin \Gamma_l^0$ *for all* $l \neq j$. *Since* Γ_l^0 *is closed, we can find an open convex cone* $\widetilde{\Gamma}_l \subset\subset \Gamma_l$ *such that*

$$\xi_0 \cdot \widetilde{\Gamma}_l < 0 \quad \text{for all } l \neq j.$$

Now, we invoke Corollary 38. Since both $WF_0^X(u_l')$ *and* $WF_0^X(u_l) \subset \Gamma_l^0$, *we can find a unified wedge* $\mathcal{W}_l$ *in* $\mathbb{C}^m$ *with edge* X *such that* $J\widetilde{\Gamma}_l \subset \Gamma_0(\mathcal{W}_l)$, *and holomorphic functions* $f_l, f_l' \in \mathcal{O}(\mathcal{W}_l)$, *of tempered growth, such that* $u_l = bf_l$ *and* $u_l' = bf_l'$ *on* X. *Hence,*

$$u_j' - u_j = \sum_{l \neq j} u_l - u_l' = \sum_{l \neq j} b(f_l - f_l') \quad \text{in } X.$$

If we set

$$f_l'' = f_l - f_l',$$

then we get that

$$f_l'' \in \mathcal{O}(\mathcal{W}_l); \quad \text{and} \quad u_j' - u_j = \sum_{l \neq j} bf_l'' \quad \text{in } X$$

proving that $\xi_0 \notin WF_0^X(u_j' - u_j)$ *and we are done.*

CHAPTER 2

Boundary Values of Solutions of Complex Vector Fields

2.1 Introduction

Let N be a submanifold of a smooth manifold M. In a neighborhood of a point of N we may introduce coordinates (x, t) for M with $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$ in which, locally, $N = \{t = 0\}$. By a wedge in M with edge N we mean an open set $\mathcal{W} \subseteq M$ which in some such coordinate system is of the form $\mathcal{W} = \mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is a ball in $\mathbb{R}^m$ and $\mathcal{C}$ is a truncated, open convex cone in $\mathbb{R}^n \backslash \{0\}$. When $(M, \mathcal{V})$ is a hypo-analytic structure, a submanifold E of M is called strongly noncharacteristic if $\mathbb{C}T_pM = \mathbb{C}T_pE + \mathcal{V}_p$ for each $p \in E$, and maximally real if $\mathbb{C}T_pM = \mathbb{C}T_pE \oplus \mathcal{V}_p$ for each $p \in E$. Suppose $\mathcal{W}$ is a wedge in M whose edge E is maximally real. Let $f \in \mathcal{D}'(\mathcal{W})$ be a solution of $\mathcal{V}$. Let (x, t) be a coordinate system in which $E = \{t = 0\}$ and $\mathcal{W} = \mathcal{B} \times \mathcal{C}$ as above. It is known that the solution f is a smooth function of $t \in \mathcal{C}$ valued in distributions in x-space $\mathcal{B}$. In this chapter, we prove a sufficient condition for the existence of a boundary value for f, bf, at $t = 0$ when f is continuous on the wedge $\mathcal{W}$. This generalizes previous results in [BH1] and [BH2]. Then we prove a similar result (see Theorem 50) when our involutive structure is not necessarily locally integrable.

2.2 Existence of Boundary Values in the Locally Integrable Case

Suppose L is a smooth complex vector field,

$$L = \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j} \tag{2.1}$$

defined on a domain $\Omega \subseteq \mathbb{R}^N$ and $f \in C(\Omega)$ is such that $Lf = 0$ in Ω. Assume $\partial\Omega$ is smooth. We would like to explore conditions on f that guarantee that f will have a distribution boundary value on $\partial\Omega$. Theorem 24 showed us that when f is holomorphic on a domain $D \subseteq \mathbb{C}^N$, then f has a boundary value if

$$|f(z)| \leq \frac{C}{\text{dist}\,(z, \partial\Omega)^k}$$

for some $C, k > 0$. For simplicity, we recall here a precise statement in the planar case:

Proposition 44 *Let $A, B > 0$, $Q = (-A, A) \times (0, B)$ and suppose that f is holomorphic in Q. If for some integer $N \geq 0$ and $C > 0$,*

$$|f(x+iy)| \leq Cy^{-N}, \qquad x + iy \in Q,$$

then there exists $bf \in \mathcal{D}'(-A, A)$ of order $N+1$ such that

$$\lim_{y \to 0+} \int f(x+iy)\psi(x)dx = \langle bf, \psi \rangle \qquad \forall \psi \in C_0^{N+1}(-A, A).$$

Because of the local equivalence of L^1 and sup norms for solutions in the elliptic (Cauchy-Riemann) case, the preceeding proposition asserts that a holomorphic function f on Q has a boundary value (trace) at $y = 0$ if for some integer $N > 0$,

$$\iint_Q |f(x+iy)|\, y^N dxdy < \infty.$$

From now on, unless we state otherwise, we shall reason under the following setup: Let $x = (x_1, ..., x_m) \in \mathbb{R}^m$, $t = (t_1, ..., t_n) \in \mathbb{R}^n$, and suppose that $U \subset \mathbb{R}^{m+n}$ is an open set, $0 \in U$, and $\Phi(x,t) : U \to \mathbb{R}^m$ is a smooth function satisfying

$$\Phi(0,0) = 0 \ \text{ and } \ \Phi_x(0,0) = 0.$$

For simplicity, suppose that $U = B_r(0) \times B_\delta(0) \subset \mathbb{R}^m \times \mathbb{R}^n$. Let

$$\begin{aligned} Z(x,t) &= x + i\Phi(x,t) \\ &= (x_1 + i\Phi_1(x,t), ..., x_m + i\Phi_m(x,t)) \\ &= (Z_1(x,t), ..., Z_m(x,t)). \end{aligned}$$

For $1 \leq k \leq m$, let M_k be the vector fields in x-space satisfying

$$M_k Z_l = \delta_{kl} \quad \text{for } 1 \leq k, l \leq m,$$

and consider the locally integrable structure $\mathbb{L} = \{L_1, ..., L_n\}$ generated by the vector fields

$$L_j = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} \frac{\partial Z_k}{\partial t_j}(x,t) M_k.$$

Note that $L_j Z_k = 0$ for all $1 \leq j \leq n$, $1 \leq k \leq m$. In Theorem 45, we will give a sufficient condition for the existence of a boundary value of a continuous function f, bf, when f is a solution of $\mathbb{L}f = 0$. In Theorem 49, we shall give a formula for bf. This generalizes previous results in [BH1] and [BH2]. Before we state the Theorems, we make some conventions:

(1) We write $\mathbb{R}^m_x$ to denote $\mathbb{R}^m$ with coordinates $x = (x_1, ..., x_m)$.

(2) We write $g(x,t) \in C^\infty_{0,x}(\mathbb{R}^m_x \times \mathbb{R}^n_t)$ if $g(x,t) \in C^\infty(\mathbb{R}^m_x \times \mathbb{R}^n_t)$ and the x-support of g is contained in a fixed compact set independent of t.

(3) We write $\Gamma_\delta \subset \mathbb{R}^n_t$ to denote an acute open convex cone $\Gamma \subset \mathbb{R}^n_t$ intersected with $B_\delta(0) \subset \mathbb{R}^n_t$.

(4) In Theorem 45, we will make use of the vector fields V_k that are the restrictions of the vector fields M_k to the maximally real submanifold $\Sigma = \{Z(x,0) = x + i\Phi(x,0) : x \in B_r(0)\}$; i.e., $V_k = M_k|_\Sigma$. Thus, $V_k\,[Z_l(x,0)] = \delta_{kl}$ for $1 \leq k, l \leq m$.

(5) Finally, if $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$ is a multi-index, then V^α will denote $V_1^{\alpha_1} \cdots V_m^{\alpha_m}$.

Theorem 45 *Let* $\mathcal{W} = B_r(0) \times \Gamma_\delta \subset \mathbb{R}^m_x \times \mathbb{R}^n_t$ *and suppose that* $f(x,t) \in C(\mathcal{W})$ *satisfies*

(1) $\int_{B_r(0)} |L_j f(x,t)|\, dx \leq C < \infty$; *and*

(2) $\exists N \in \mathbb{N}$ *such that*

$$|f(x,t)|\, |Z(x,t) - Z(x,0)|^N \leq C < \infty.$$

Then $bf = \lim_{\Gamma_\delta \ni t \to 0} f(.,t)$ *exists in* $D'(B_r(0))$.

Proof. Let

$$P(x,t) = \Phi(x,t) - \Phi(x,0).$$

For $g(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$ and for $k = 0, 1, 2, ...$, define

$$(T_k g)(x,t) = \sum_{|\alpha| \leq k} \frac{i^{|\alpha|}}{\alpha!} (V^\alpha g)(x,t) P^\alpha(x,t).$$

Note that

$$(T_0 g)(x,t) = g(x,t).$$

Fix $\psi \in C^\infty_0(B_r(0))$. We will divide the proof into 3 steps:

Step (1): We claim that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\, (T_N \psi)(x,t) dZ(x,t) \;\; \text{exists} \qquad \text{(See Remark 46)}$$

Now, for a general $g(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$, existence of the above limit for an arbitrary $\psi \in C^\infty_0(B_r(0))$ implies that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\, (T_N g)(x,t) dZ(x,t) \;\; \text{exists.}$$

Step (2): We now claim that existence of the last limit implies that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\, (T_{N-1} g)(x,t) dZ(x,t) \;\; \text{exists} \qquad \text{(See Remark 47)}$$

Step (3): Finally, we claim that, in fact, for any $g(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$ and for $0 \leq k \leq N$,

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\, (T_k g)(x,t) dZ(x,t) \;\; \text{exists} \qquad \text{(See Remark 48)}$$

In particular, for $k=0$ and $g(x,t)=\psi(x)\in C_0^\infty(B_r(0))$, we get that

$$\lim_{\Gamma_\delta \ni t\to 0} \int\limits_{B_r(0)} f(x,t)\psi(x)dZ(x,t) \text{ exists}$$

Note that on the submanifold $B_r(0)\times\{t_0\}$, we have

$$dZ(x,t_0)=Z_x(x,t_0)dx,$$

where

$$Z_x(x,t)=I+i\Phi_x(x,t).$$

Thus,

$$\lim_{\Gamma_\delta \ni t\to 0} \int\limits_{B_r(0)} f(x,t)\psi(x)dZ(x,t) = \lim_{\Gamma_\delta \ni t\to 0} \int\limits_{B_r(0)} f(x,t)Z_x(x,t)\psi(x)dx = \langle Z_x(x,0)bf,\psi\rangle .$$

This shows that $bf=\lim\limits_{\Gamma_\delta \ni t\to 0} f(.,t)$ exists in $D'(B_r(0))$. ■

Remark 46 *For $k=0,1,2,...$ define*

$$u_k(x,y)=\sum_{|\alpha|\le k} \frac{i^{|\alpha|}}{\alpha!}\left(V^\alpha\psi\right)(x)\left(y-\Phi(x,0)\right)^\alpha .$$

We claim that:

(a) $u_k(Z(x,0))=\psi(x)$; *and*
(b) $\left|\frac{\partial u_k}{\partial \bar{z}_j}(x,y)\right|\le C\,dist((x,y),\Sigma)^k$ *for some $C>0$ and all $k\le N$.*

Assume for the moment that the claims are true. Then we would get:

(i) $\left(T_k\psi\right)(x,0)=\psi(x)$; *and*
(ii) $\left|L_j\left(T_k\psi\right)(x,t)\right|\le C\left|Z(x,t)-Z(x,0)\right|^k$ *for all $1\le j\le n$.*
To see this, note that

$$\left(T_k\psi\right)(x,t)=u_k(Z(x,t)),$$

and so, by (a),

$$\left(T_k\psi\right)(x,0)=u_k(Z(x,0))=\psi(x).$$

Also,

$$\begin{aligned} L_j\left(T_k\psi\right)(x,t) &= L_j\left(u_k(Z(x,t))\right) \\ &= L_j\left(u_k(x,\Phi(x,t))\right) \\ &= \sum_{l=1}^{m}\left(\frac{\partial u_k}{\partial x_l}(x,\Phi(x,t))L_j(x_l)+\frac{\partial u_k}{\partial y_l}(x,\Phi(x,t))L_j(\Phi_l(x,t))\right) \\ &= 2\sum_{l=1}^{m}\frac{\partial u_k}{\partial \bar{z}_l}(x,\Phi(x,t))L_j(x_l), \end{aligned}$$

where the last equality follows since $L_jZ_l(x,t)=0$ *and so* $L_j(\Phi_l(x,t))=iL_j(x_l)$*. Hence, Using (b), (ii) follows. We will now show the validity of claims (a) and (b). We have*

$$u_k(Z(x,0))=u_k(x,\Phi(x,0))=\sum_{|\alpha|\le k}\frac{i^{|\alpha|}}{\alpha!}\left(V^\alpha\psi\right)(x)\left(\Phi(x,0)-\Phi(x,0)\right)^\alpha=\psi(x).$$

This proves (a). To see why (b) is true, we will prove, using induction on k*, that*

$$2\frac{\partial u_k}{\partial \bar{z}_l}(x,y)=i^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\frac{\partial}{\partial x_l}\left(V^\alpha\psi\right)(x)\left(y-\Phi(x,0)\right)^\alpha \tag{2.2}$$

For $k=1$*,*

$$u_1(x,y)=\psi(x)+i\sum_{s=1}^{m}\left(V_s\psi\right)(x)\left(y_s-\Phi_s(x,0)\right),$$

and so,

$$\begin{aligned} \frac{\partial u_1}{\partial y_l}(x,y) &= i\left(V_l\psi\right)(x);\ \text{and} \\ \frac{\partial u_1}{\partial x_l}(x,y) &= \frac{\partial \psi}{\partial x_l}(x)+i\sum_{s=1}^{m}\frac{\partial}{\partial x_l}\left(V_s\psi\right)(x)\left(y_s-\Phi_s(x,0)\right)-i\sum_{s=1}^{m}\left(V_s\psi\right)(x)\frac{\partial \Phi_s}{\partial x_l}(x,0). \end{aligned}$$

Next, observe that

$$\frac{\partial}{\partial x_l}=V_l+i\sum_{s=1}^{m}\frac{\partial \Phi_s}{\partial x_l}(x,0)V_s \tag{2.3}$$

Thus, using (2.3)*, we get* (2.2) *for* $k=1$*. Now, suppose that* (2.2) *holds for* $k-1,\ k\ge 1$*. We can write*

$$u_k(x,y)=u_{k-1}(x,y)+E_k(x,y),$$

where

$$E_k(x,y)=i^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\left(V^\alpha\psi\right)(x)\left(y-\Phi(x,0)\right)^\alpha.$$

Using the induction hypothesis on $u_{k-1}(x,y)$ *and* (2.3), *we can write*

$$2\frac{\partial u_{k-1}}{\partial \overline{z}_l}(x,y) = i^k \sum_{|\beta|=k-1} \sum_{s=1}^{m} \frac{1}{\beta!}\frac{\partial \Phi_s}{\partial x_l}(x,0)V_s\left(\left(V^\beta\psi\right)(x)\right)(y-\Phi(x,0))^\beta + i^{k-1}\sum_{|\beta|=k-1}\frac{1}{\beta!}V_l\left(\left(V^\beta\psi\right)(x)\right)(y-\Phi(x,0))^\beta .$$

Now, we can easily obtain that

$$2\frac{\partial E_k}{\partial \overline{z}_l}(x,y) = i^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\left[\frac{\partial}{\partial x_l}\left(V^\alpha\psi\right)(x)(y-\Phi(x,0))^\alpha + \left(V^\alpha\psi\right)(x)\frac{\partial}{\partial x_l}(y-\Phi(x,0))^\alpha\right] - i^{k-1}\sum_{|\alpha|=k}\frac{1}{\alpha!}\left(V^\alpha\psi\right)(x)\frac{\partial}{\partial y_l}(y-\Phi(x,0))^\alpha .$$

Hence, adding the last two equations, we get

$$2\frac{\partial u_k}{\partial \overline{z}_l}(x,y) = 2\frac{\partial u_{k-1}}{\partial \overline{z}_l}(x,y) + 2\frac{\partial E_k}{\partial \overline{z}_l}(x,y) = i^k\sum_{|\alpha|=k}\frac{1}{\alpha!}\frac{\partial}{\partial x_l}\left(V^\alpha\psi\right)(x)(y-\Phi(x,0))^\alpha .$$

This ends the proof of claim (b). Recall that the main purpose of Remark 46 is to prove the existence of

$$\lim_{\Gamma_\delta \ni t\to 0}\int_{B_r(0)} f(x,t)\left(T_N\psi\right)(x,t)dZ(x,t).$$

For this, note that for any C^1 *function* $g(x,t)$ *defined near the origin in* $\mathbb{R}^m\times\mathbb{R}^n$,

$$dg(x,t) = \sum_{j=1}^{n} L_j g(x,t)dt_j + \sum_{k=1}^{m} M_k g(x,t)dZ_k(x,t).$$

Consider the m-form

$$\omega(x,t) = g(x,t)dZ(x,t).$$

Then

$$d\omega = d\left(gdZ\right) = dg\wedge dZ = \sum_{j=1}^{n} L_j g dt_j\wedge dZ.$$

Plugging

$$g(x,t) = f(x,t)\left(T_N\psi\right)(x,t),$$

we get that

$$d\omega = \sum_{j=1}^{n} f(x,t)L_j\left(T_N\psi\right)(x,t)dt_j\wedge dZ + \sum_{j=1}^{n} L_j f(x,t)\left(T_N\psi\right)(x,t)dt_j\wedge dZ.$$

Fix $T\in\Gamma_\delta$ *and let* $\delta' = \delta - |T|$. *For* $s\in\Gamma_{\delta'}$, *define*

$$\gamma_s(\tau) = (1-\tau)s + \tau T.$$

We now avail ourselves of Stokes Theorem:

$$\int_{B_r(0)} \int_{\gamma_s} d\omega(x,t) = \int_{B_r(0)} \omega(x,T) - \int_{B_r(0)} \omega(x,s).$$

Writing things out explicitly, we get

$$\begin{aligned} \int_{B_r(0)} f(x,s)\left(T_N\psi\right)(x,s)dZ(x,s) &= \int_{B_r(0)} f(x,T)\left(T_N\psi\right)(x,T)dZ(x,T) \\ &\quad - \sum_{j=1}^{n} \int_{B_r(0)} \int_{\gamma_s} L_j f(x,t)\left(T_N\psi\right)(x,t)dt_j \wedge dZ(x,t) \\ &\quad - \sum_{j=1}^{n} \int_{B_r(0)} \int_{\gamma_s} f(x,t) L_j\left(T_N\psi\right)(x,t)dt_j \wedge dZ(x,t) \end{aligned} \tag{2.4}$$

The first integral on the RHS clearly exists. The second integral on the RHS exists, independently of s, by assumption (1) of the theorem. Now, since

$$\left|L_j\left(T_N\psi\right)(x,t)\right| \le C\left|Z(x,t) - Z(x,0)\right|^N \qquad \textit{for all } 1 \le j \le n,$$

and by assumption (2) of the theorem, we get that the third integral on the RHS exists, independently of s, and hence $\lim_{\Gamma_\delta \ni s \to 0} \int_{B_r(0)} f(x,s)\left(T_N\psi\right)(x,s)dZ(x,s)$ exists as well.

Remark 47 *Here, we are assuming that*

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\left(T_N g\right)(x,t)dZ(x,t) \quad \textit{exists,}$$

and we want to show that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\left(T_{N-1} g\right)(x,t)dZ(x,t) \quad \textit{exists.}$$

To do so, suppose that

$$g(x,t) = \psi(x,t)P(x,t)^\beta,$$

for some $\psi(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$ and for a multi-index β with $|\beta| = N$. Note that we may write

$$T_N\left(\psi P^\beta\right)(x,t) = \psi(x,t)P(x,t)^\beta + \psi(x,t) \sum_{|\alpha|=N} e_\alpha(x,t)P(x,t)^\alpha + \sum_{|\gamma|>N} h_\gamma(x,t)P(x,t)^\gamma \tag{2.5}$$

where $e_\alpha(x,t)$ *and* $h_\gamma(x,t)$ *are smooth functions and*

$$\lim_{t\to 0} D_x^{\alpha'} e_\alpha(x,t) = 0,$$

for all multi-indices α, α'. *Also, by our assumption on the growth of* f *and the fact that*

$$|P(x,t)| = |Z(x,t) - Z(x,0)|,$$

we get that for each multi-index γ *with* $|\gamma| > N$,

$$\lim_{\Gamma_\delta \ni t \to 0} \int\limits_{B_r(0)} f(x,t) h_\gamma(x,t) P(x,t)^\gamma dZ(x,t) \quad \textit{exists.}$$

Using (5), *we get that*

$$\lim_{\Gamma_\delta \ni t \to 0} \int\limits_{B_r(0)} f(x,t) \left(\psi(x,t) P(x,t)^\beta + \psi(x,t) \sum_{|\alpha|=N} e_\alpha(x,t) P(x,t)^\alpha \right) dZ(x,t) \quad \textit{exists.}$$

Since $\psi(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$ *was chosen arbitrarily, we can substitute* $\psi_\beta(x,t)$ *for* $\psi(x,t)$ *in the last limit, where* $\psi_\beta(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$ *and sum over* β *with* $|\beta| = N$, *to conclude that*

$$\lim_{\Gamma_\delta \ni t \to 0} \int\limits_{B_r(0)} f(x,t) \left(\sum_{|\beta|=N} \psi_\beta P^\beta \left(1 + E_\beta(x,t)\right) \right) dZ(x,t) \quad \textit{exists.}$$

where $\lim_{t\to 0} D_x^{\beta'} E_\beta(x,t) = 0$ *for all multi-indices* β'. *It follows that*

$$\lim_{t\to 0} \sum_{|\beta|=N} \psi_\beta P^\beta \left(1 + E_\beta(x,t)\right) = \sum_{|\beta|=N} \psi_\beta P^\beta \quad \textit{in } C^\infty_0(B_r(0)).$$

This implies that

$$\lim_{\Gamma_\delta \ni t \to 0} \int\limits_{B_r(0)} f(x,t) \left(\sum_{|\beta|=N} \psi_\beta P^\beta \right) dZ(x,t) \ \textit{exists, whenever } \psi_\beta(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0)).$$

Now, note that for $g(x,t) \in C^\infty_{0,x}(B_r(0) \times B_\delta(0))$,

$$\left(T_N g\right)(x,t) = \left(T_{N-1} g\right)(x,t) + \sum_{|\beta|=N} \psi_\beta(x,t) P(x,t)^\beta,$$

where

$$\psi_\beta(x,t) = \frac{i^{|\beta|}}{\beta!} \left(V^\beta g\right)(x,t).$$

Hence, $\lim\limits_{\Gamma_\delta \ni t \to 0} \int\limits_{B_r(0)} f(x,t) \left(T_{N-1} g\right)(x,t) dZ(x,t)$ *exists.*

Remark 48 *We use descending induction. The proof is identical to that in Remark 47 but with appropriate modifications.*

We avail ourselves of the proof of Theorem 45 to get a formula for bf :

Theorem 49 *Under the hypotheses and notation of Theorem 45, we have the following formula for* bf : *For any* $\psi \in C_0^\infty(B_r(0))$,

$$\begin{aligned} \langle Z_x(x,0)bf,\psi\rangle &= \int\limits_{B_r(0)} f(x,T)\left(T_N\psi\right)(x,T)dZ(x,T) \\ &\quad -\sum_{j=1}^{n}\int\limits_{B_r(0)}\int\limits_{\gamma_0} L_jf(x,t)\left(T_N\psi\right)(x,t)dt_j\wedge dZ(x,t) \\ &\quad -\sum_{j=1}^{n}\int\limits_{B_r(0)}\int\limits_{\gamma_0} f(x,t)L_j\left(T_N\psi\right)(x,t)dt_j\wedge dZ(x,t). \end{aligned}$$

(Here, γ_0 *is the line segment joining* 0 *to* T*). This formula shows that* bf *is a distribution of order* $N+1$.

Proof. We have established the existence of $\lim\limits_{\Gamma_\delta\ni s\to 0}\int\limits_{B_r(0)} f(x,s)\left(T_N\psi\right)(x,s)dZ(x,s)$ in Theorem 45 and we showed that it equals to the RHS of the formula in the statement of this theorem. Hence, we will be done if we can show that this limit is equal to $\langle Z_x(x,0)bf,\psi\rangle$. This follows since the functions

$$x\longrightarrow\left(T_N\psi\right)(x,s)-\psi(x)\qquad\text{and}\qquad x\longrightarrow Z(x,s)-Z(x,0)$$

and all their x-derivatives converge to 0 as $s\to 0$ and so

$$Z_x(x,s)\left(T_N\psi\right)(x,s)\longrightarrow Z_x(x,0)\psi(x)$$

as $s\to 0$ in $C_0^\infty(B_r(0))$. ■

2.3 Existence of Boundary Values in General

Suppose $\mathcal{V} = \{L_1, ..., L_n\}$ is a system of smooth complex vector fields

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk}(x,t)\frac{\partial}{\partial x_k}$$

in a neighborhood U of the origin in $\mathbb{R}^m_x \times \mathbb{R}^n_t$. For simplicity, say $U = B_r(0) \times B_\delta(0)$ and let $\mathcal{W} = B_r(0) \times \Gamma_\delta$ be a wedge where $\Gamma_\delta \subset \mathbb{R}^n_t$ is a truncated open convex cone. For analogues of the following theorem for a single vector field see Theorem 1.1 in [BH] and Theorem VI.1.3 in [BCH]:

Theorem 50 *Let $\mathcal{W} = B_r(0) \times \Gamma_\delta$ be as above and suppose that $f(x,t) \in C(\mathcal{W})$ satisfies: for some $C > 0$ and some $N \in \mathbb{N}$*

(i)

$$\int_{B_r(0)} |L_j f(x,t)|\, dx \leq C$$

and (ii)

$$|f(x,t)|\, |t|^N \leq C.$$

Then $bf = \lim_{\Gamma_\delta \ni t \to 0} f(.,t)$ exists in $\mathcal{D}'(B_r(0))$.

Proof. Let $Z_1, ..., Z_m : U \to \mathbb{C}$ be a complete set of smooth approximate first integrals for $\mathcal{V}$ near the origin in U. That is,

$$L_j Z_k(x,t) = O(|t|^l) \text{ for } l = 1, 2, ..., \text{ and } Z_k(x,0) = x_k, \quad 1 \leq k \leq m. \tag{2.6}$$

Define

$$b_{jk}(x,t) = L_j Z_k(x,t). \tag{2.7}$$

Write

$$\begin{aligned} Z(x,t) &= (Z_1(x,t), ..., Z_m(x,t)); \text{ and} \\ Z_k(x,t) &= \Psi_{1k}(x,t) + i\Psi_{2k}(x,t), \end{aligned}$$

where $\Psi_{1k}(x,t)$ and $\Psi_{2k}(x,t)$ are real-valued. For $j = 1, ..., m$, let

$$M_j = \sum_{k=1}^{m} c_{jk}(x,t)\frac{\partial}{\partial x_k}$$

be vector fields in x-space satisfying

$$M_j Z_k = \delta_{jk}, \qquad [M_j, M_k] = 0. \tag{2.8}$$

Note that for each j, k,

$$[M_j, L_k] = \sum_{l=1}^{m} d_{jkl}(x,t) M_l, \tag{2.9}$$

where each $d_{jkl}(x,t) = O(|t|^s)$ for $s = 1, 2, ...$ Indeed, the latter can be seen by expressing $[M_j, L_k]$ in terms of the basis $\{L_1, ..., L_n, M_1, ..., M_m\}$ and applying both sides to the $n+m$ functions $\{t_1, ..., t_n, Z_1, ..., Z_m\}$. Equations (2.6) and (2.7) imply that

$$M_k b_{jk} = O(|t|^s) \ \text{ for } s = 1, 2, ... \tag{2.10}$$

Using (2.7) and (2.8), we obtain

$$L_j \Psi_{2k} = iL_j \Psi_{1k} - ib_{jk} \tag{2.11}$$

$$M_j \Psi_{2k} = iM_j \Psi_{1k} - i\delta_{jk}. \tag{2.12}$$

Now, if $g(x,t)$ is any C^1 function defined in U, observe that the differential

$$dg = \sum_{k=1}^{m} M_k(g)\, dZ_k + \sum_{j=1}^{n} L_j(g)\, dt_j - \sum_{j=1}^{n}\sum_{k=1}^{m} M_k(g)\, b_{jk} dt_j. \tag{2.13}$$

Hence, if we consider the m-form $\omega = g\, dZ$, we get

$$d\omega = dg \wedge dZ = \sum_{j=1}^{n} L_j(g)\, dt_j \wedge dZ - \sum_{j=1}^{n}\sum_{k=1}^{m} M_k(g)\, b_{jk}\, dt_j \wedge dZ. \tag{2.14}$$

Observe that hypothesis (ii) in the theorem together with the fact that

$$b_{jk}(x,t) = O(|t|^s) \quad M_k b_{jk}(x,t) = O(|t|^s) \quad \forall s$$

imply that $\forall \varphi \in C_0^\infty(B_r(0))$,

$$\left| \int_{\Gamma_\delta} \int_{B_r(0)} b_{jk}(x,t) M_k f(x,t) \varphi(x) dx dt \right| \le C_2, \tag{2.15}$$

where $C_2 > 0$ is a constant that depends only on $\sup\limits_{x \in B_r(0)} \sum_{|\alpha| \le 1} \|D^\alpha \varphi(x)\|$. Let

$$\Psi_1 = (\Psi_{11}, \ldots, \Psi_{1m}) \quad \text{and} \quad \Psi_2 = (\Psi_{12}, \ldots, \Psi_{2m}).$$

For $\varphi \in C_0^\infty(B_r(0))$ and k a nonnegative integer, define

$$T_k\varphi(x,t) = \sum_{|\alpha|\le k} \frac{i^{|\alpha|}}{\alpha!} \left[\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(\Psi_1(x,t))\right] (\Psi_2(x,t))^\alpha . \tag{2.16}$$

We will first show that $\lim_{\Gamma_\delta \ni t\to 0} \int_{B_r(0)} f(x,t)\,(T_N\varphi)\,(x,t)dZ(x,t)$ exists. To prove this, fix $T \in \Gamma_\delta$ and let $\delta' = \delta - |T|$. For $s \in \Gamma_{\delta'}$, define $\gamma_s(\tau) = (1-\tau)\,s + \tau T$, $0 \le \tau \le 1$. Let $\omega = (fT_N\varphi)\,dZ$. Using (2.14) and Stokes' theorem, we get

$$\begin{aligned}\int_{B_r(0)} f(x,s)\quad (T_N\varphi)\,(x,s)\,dZ(x,s) &= \int_{B_r(0)} f(x,T)\,(T_N\varphi)\,(x,T)\,dZ(x,T)\\ &-\sum_{j=1}^n \int_{B_r(0)}\int_{\gamma_s}\left(L_jf - \sum_{k=1}^m M_k\,(f)\,b_{jk}\right) T_N\varphi dt_j\wedge dZ\\ &-\sum_{j=1}^n \int_{B_r(0)}\int_{\gamma_s}\left(L_jT_N\varphi - \sum_{k=1}^m M_k\,(T_N\varphi)\,b_{jk}\right) f dt_j\wedge dZ. \end{aligned}\tag{2.17}$$

The second integral on the RHS has a limit as $s \to 0$ by hypothesis (i) of the theorem and the argument similar to the one used to get (2.15). For the third integral, consider

$$\begin{aligned} L_jT_N\varphi &= \sum_{|\alpha|\le N} \frac{i^{|\alpha|}}{\alpha!}\left[L_j\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(\Psi_1)\right](\Psi_2)^\alpha\\ &+ \sum_{|\alpha|\le N} \frac{i^{|\alpha|}}{\alpha!}\left[\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(\Psi_1)\right] L_j\,(\Psi_2)^\alpha\\ &= \sum_{|\alpha|\le N}\sum_{l=1}^m \frac{i^{|\alpha|}}{\alpha!}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha+e_l}\varphi(\Psi_1)\right)(L_j\Psi_{1l})\,(\Psi_2)^\alpha\\ &+ \sum_{1\le|\alpha|\le N}\sum_{l=1}^m \frac{i^{|\alpha|}}{\alpha!}\left(\frac{\partial}{\partial x}\right)^{\alpha}\varphi(\Psi_1)\left[\alpha_l\,(\Psi_2)^{\alpha-e_l}\,L_j\Psi_{2l}\right]\\ &= \sum_{|\alpha|\le N}\sum_{l=1}^m \frac{i^{|\alpha|}}{\alpha!}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha+e_l}\varphi(\Psi_1)\right)(L_j\Psi_{1l})\,(\Psi_2)^\alpha\\ &+ \sum_{|\alpha|\le N-1}\sum_{l=1}^m \frac{i^{|\alpha|+1}}{\alpha!}\left(\frac{\partial}{\partial x}\right)^{\alpha+e_l}\varphi(\Psi_1)\,(L_j\Psi_{2l})\,(\Psi_2)^\alpha\\ &= \sum_{|\alpha|= N}\sum_{l=1}^m \frac{i^{|\alpha|}}{\alpha!}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha+e_l}\varphi(\Psi_1)\right)(L_j\Psi_{1l})\,(\Psi_2)^\alpha\\ &+ \sum_{|\alpha|\le N-1}\sum_{l=1}^m \frac{i^{|\alpha|}}{\alpha!}\left(\frac{\partial}{\partial x}\right)^{\alpha+e_l}\varphi(\Psi_1)\,(\Psi_2)^\alpha\,L_j(Z_l). \end{aligned}\tag{2.18}$$

Since $Z(x,0)=x$, $|\Psi_2(x,t)| = |\Psi_2(x,t)-\Psi_2(x,0)| \le C' |t|$ and so, recalling that the Z_l are approximate solutions, we conclude that

$$|L_j T_N \varphi(x,t)| \le C'_j |t|^N . \tag{2.19}$$

Hence,

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t) T_N \varphi(x,t) dZ(x,t) \quad \text{exists.}$$

We will next use the existence of

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t) \left(T_N g\right)(x,t) dZ(x,t)$$

to show that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t) \left(T_{N-1} g\right)(x,t) dZ(x,t) \quad \text{exists.}$$

To do so, let $\psi(x,t) \in C_0^\infty(B_r(0) \times B_\delta(0))$ and for a fixed multi-index β with $|\beta| = N$ let

$$g(x,t) = \tilde{\psi}(x,t) \widetilde{\Psi}_2(x,t)^\beta,$$

where $\tilde{\psi}(x,t) = \psi(\Psi_1(x,t),t)$ and $\widetilde{\Psi}_2(x,t) = \Psi_2(\Psi_1(x,t),t)$. The functions $\tilde{\psi}$ and $\widetilde{\Psi}_2(x,t)$ exist since the map $(x,t) \to (\Psi_1(x,t),t)$ is a diffeomorphism. Note that we may write

$$\begin{aligned} T_N\left(\tilde{\psi}\widetilde{\Psi}_2^\beta\right)(x,t) &= \psi(x,t)\Psi_2(x,t)^\beta + \psi(x,t) \sum_{|\alpha|=N} a_\alpha(x,t)\Psi_2(x,t)^\alpha \\ &+ \sum_{|\gamma|>N} b_\gamma(x,t)\Psi_2(x,t)^\gamma \end{aligned} \tag{2.20}$$

where $a_\alpha(x,t)$ and $b_\gamma(x,t)$ are smooth and $a_\alpha(x,0) \equiv 0$. The assumption on the growth of f implies that

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t) \left(\psi(x,t) \sum_{|\alpha|=N} a_\alpha(x,t)\Psi_2(x,t)^\alpha + \sum_{|\gamma|>N} b_\gamma(x,t)\Psi_2(x,t)^\gamma \right) dZ$$

exists. It follows that for any $\psi(x,t) \in C_0^\infty(B_r(0) \times B_\delta(0))$ and any multi-index β with $|\beta| = N$,

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\psi(x,t)\Psi_2(x,t)^\beta \, dZ(x,t)$$

exists. Note next that for any $g(x,t) \in C_0^\infty(B_r(0) \times B_\delta(0))$,

$$T_N g(x,t) = T_{N-1} g(x,t) + \sum_{|\beta|=N} \psi_\beta(x,t)\Psi_2(x,t)^\beta$$

for some smooth ψ_β of compact support. Hence,

$$\lim_{\Gamma_\delta \ni t \to 0} \int_{B_r(0)} f(x,t)\,(T_{N-1}g)\,(x,t) dZ(x,t)$$

exists. We will prove by descending induction that for any
$g(x,t) \in C_0^\infty(B_r(0) \times B_\delta(0))$ and $0 \le k \le N$,

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_k g(x,t)\, dZ(x,t) \quad \text{ exists,}$$

which for $k = 0$ and $g(x,t) = \psi(x) \in C_0^\infty(B_r(0))$ proves the Theorem. To proceed by induction, suppose $1 \le k \le N$ and assume that for any multi-index β with $|\beta| = k$, the limits

$$\begin{aligned} &\lim_{t\to 0} \int_{B_r(0)} f(x,t) \Psi_2(x,t)^\beta g(x,t)\, dZ(x,t) \quad \text{and} \\ &\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_{k-1} g(x,t)\, dZ(x,t) \end{aligned} \tag{2.21}$$

both exist for any $g(x,t) \in C_0^\infty(B_r(0) \times B_0(r))$. We have already seen that (2.21) is true for $k = N$. Fix β' with $|\beta'| = k-1$. Plug $g(x,t) = \tilde{\psi}(x,t)\widetilde{\Psi}_2(x,t)^{\beta'}$ in the limit on the right in (2.21) and observe that $T_{k-1}g$ may be written as

$$T_{k-1}g(x,t) = \psi(x,t)\Psi_2(x,t)^{\beta\prime} + \psi(x,t) \sum_{|\alpha|=k-1} c_\alpha(x,t)\Psi_2(x,t)^\alpha + \sum_{|\gamma|\ge k} d_\gamma(x,t)\Psi_2(x,t)^\gamma \tag{2.22}$$

where $c_\alpha(x,t)$ and $d_\gamma(x,t)$ are smooth and $c_\alpha(x,0) \equiv 0$. From the existence of the two limits in (2.21) we derive that

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t)(\psi(x,t)\Psi_2(x,t)^{\beta\prime} + \psi(x,t) \sum_{|\alpha|=k-1} c_\alpha(x,t)\Psi_2(x,t)^\alpha)\, dZ(x,t) \tag{2.23}$$

exists. Observe next that since each $c_\alpha(x,0) \equiv 0$, given any collection $\{\psi_\beta(x,t) : |\beta| = k-1\}$ of compactly supported functions, we can find compactly supported functions $\{\eta_{\beta'}(x,t) : |\beta'| = k-1\}$ such that

$$\sum_{\beta'} \eta_{\beta'} \Psi_2^{\beta'} + \sum_{\beta'} \eta_{\beta'} (\sum c_\alpha \Psi_2^\alpha = \sum \psi_\beta \Psi_2^\beta.$$

We conclude that

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t)\Psi_2(x,t)^\beta \psi(x,t)\, dZ(x,t) \quad \text{ exists} \tag{2.24}$$

for all β with $|\beta| = k-1$ and $\psi(x,t) \in C^\infty(B_r(0)\times B_r(0))$. Hence, taking account of (2.21) and (2.24) we conclude that

$$\lim_{t\to 0}\int_{B_r(0)} f(x,t)T_{k-2}g(x,t)\,dZ(x,t) \quad \text{exists.} \tag{2.25}$$

We have thus proved that (2.21) holds for $k-1$, completing the inductive step. Therefore,

$$\lim_{t\to 0}\int_{B_r(0)} f(x,t)\psi(x)\,dZ(x,t) \quad \text{exists} \tag{2.26}$$

and thus $bf = \lim_{t\to 0} f(.,t)$ exists. ■

For the rest of this section, let $(M,\mathcal{V})$ be $\mathbb{R}^{m+n} = \mathbb{R}^m_x \times \mathbb{R}^n_t$ with a CR structure $\mathcal{V}$ near the origin; i.e., $\mathcal{V}\cap\overline{\mathcal{V}} = \{0\}$ in a neighborhood $U = B_r(0)\times B_\delta(0)$ of the origin in $\mathbb{R}^m_x\times\mathbb{R}^n_t$. Suppose that $\mathcal{V}$ is generated, in U, by the complex vector fields $\{L_1,...,L_n\}$, where

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^m a_{jk}(x,t)\frac{\partial}{\partial x_k}.$$

Let $Z_1,...,Z_m : U \to \mathbb{C}$ be a complete set of smooth approximate first integrals for $\mathcal{V}$ in U such that

$$Z_l(x,0) = x_l, \quad 1\le l\le m.$$

For each $l = 1,...,m$, we may write

$$Z_l(x,t) = x_l + \sum_{s=1}^n t_s\psi_{ls}(x,t), \tag{2.27}$$

where $\psi_{ls}(x,t) = \psi^{(1)}_{ls}(x,t) + i\psi^{(2)}_{ls}(x,t)$. Since $\mathcal{V}$ is CR in U, for each $1\le j\le n$ there exists $1\le j'\le m$ such that

$$\Im a_{jj'}(0,0) \neq 0. \tag{2.28}$$

Observe that

$$\Im a_{jl}(0,0) = -\psi^{(2)}_{lj}(0,0). \tag{2.29}$$

Indeed,

$$\begin{aligned} L_jZ_l(x,t) &= \frac{\partial Z_l}{\partial t_j}(x,t) + \sum_{k=1}^m a_{jk}(x,t)\frac{\partial Z_l}{\partial x_k}(x,t) \\ &= \left(\sum_{s=1}^n t_s\frac{\partial\psi_{ls}}{\partial t_j}(x,t) + \psi_{lj}(x,t)\right) \\ &+ \left(\sum_{k=1}^m a_{jk}(x,t)\left(\delta_{kl} + \sum_{s=1}^n t_s\frac{\partial\psi_{ls}}{\partial x_k}(x,t)\right)\right) \end{aligned}$$

Evaluating this at $(0,0)$, we get

$$0=\psi_{lj}(0,0)+a_{jl}(0,0).$$

Corollary 51 *Let $\mathcal{W}=B_r(0)\times\Gamma_\delta$ be a wedge with edge $B_r(0)$, where $\Gamma\subset\mathbb{R}^n_t$ is an open cone with vertex at the origin, and suppose that $f(x,t)\in C(\mathcal{W})$ satisfies: for some $C>0$ and some $N\in\mathbb{N}$*

(i) $\int_{B_r(0)}|L_jf(x,t)|\,dx\leq C<\infty$; and

(ii) there exist $N\in\mathbb{N}$ and $C>0$ such that

$$|f(x,t)|\,|Z(x,t)-Z(x,0)|^N\leq C.$$

Then $bf=\lim\limits_{\Gamma_\delta\ni t\to 0}f(.,t)$ exists in $\mathcal{D}'(B_r(0))$.

Proof. If we set

$$\begin{aligned}
Z(x,t) &= (Z_1(x,t),...,Z_m(x,t)),\\
x &= (x_1,...,x_m),\\
t &= (t_1,...,t_n),\ \text{and}\\
A(x,t) &= \left(\psi_{ij}(x,t)\right)_{1\leq i\leq m,\ 1\leq j\leq n}.
\end{aligned}$$

Then we can rewrite (2.27) in the matrix form

$$Z(x,t)=x+A(x,t)t.$$

Since $\mathcal{V}$ is CR in U, $\Im A(x,t)$ has rank n at and hence near the origin. Without loss of generality, suppose that

$$B(x,t)=\left(\Im\psi_{ij}(x,t)\right)_{1\leq i,j\leq n}\ \text{is invertible near the origin.}$$

Then

$$|A(x,t)t|\geq|B(x,t)t|\geq|B_l(x,t)\cdot t|\quad\text{for all }\ (x,t)\ \text{ near }\ (0,0)\,,$$

where $B_l(x,t)$ is the l-th row of $B(x,t)$. Fix $t^0\in\Gamma$. Since $B(0,0)$ is invertible, one can find a row $B_l(0,0)$ of $B(0,0)$ such that

$$\left|B_l(0,0)\cdot\frac{t^0}{|t^0|}\right|=C_0>0.$$

Hence, we can find an open convex cone $\widetilde{\Gamma} \subset\subset \Gamma$ containing t^0 such that

$$\left| B_l(0,0) \cdot \frac{t}{|t|} \right| \geq \frac{1}{2} C_0 \quad \text{for all } t \in \widetilde{\Gamma}.$$

Therefore, we can find a wedge $\widetilde{\mathcal{W}} = B_{\widetilde{r}}(0) \times \widetilde{\Gamma}_\delta \subset\subset \mathcal{W}$ (where $0 < \widetilde{r} < r$) such that

$$\left| B_l(x,t) \cdot \frac{t}{|t|} \right| \geq \frac{1}{4} C_0 \quad \text{for all } (x,t) \in \widetilde{\mathcal{W}}.$$

This implies that for all $(x,t) \in \widetilde{\mathcal{W}}$

$$|Z(x,t) - Z(x,0)| = |A(x,t)t| \geq \frac{1}{4} C_0 \, |t| \, .$$

Thus,

$$|f(x,t)| \, |t|^N \leq \text{const.} \; |f(x,t)| \, |Z(x,t) - Z(x,0)|^N \leq C.$$

Hence, by Theorem 4.1, $bf = \lim_{\Gamma_\delta \ni t \to 0} f(.,t)$ exists in $\mathcal{D}'(B_r(0))$. ∎

CHAPTER 3

Edge of the Wedge Theory in Involutive Structures

3.1 Introduction

Let M be a C^∞ manifold and $\mathcal{V} \subseteq \mathbb{C}TM$ a subbundle with rank n which is involutive, that is, the bracket of two smooth sections of $\mathcal{V}$ is also a section of $\mathcal{V}$. We will refer to the pair $(M, \mathcal{V})$ as an involutive structure. The involutive structure $(M, \mathcal{V})$ is called locally integrable if the orthogonal of $\mathcal{V}$ in $\mathbb{C}T^*M$ is locally generated by exact forms. In [EG] assuming that $(M, \mathcal{V})$ is locally integrable, the authors proved some microlocal regularity results for a distribution u on certain submanifolds E of M where u arises as the boundary value of a solution on a wedge $\mathcal{W}$ in M with edge E. These results were expressed in terms of the hypo-analytic wave-front set developed in [BCT]. In this chapter we prove some analogous results in the setting of involutive structures that are not necessarily locally integrable, and for boundary values of approximate solutions (Definition 57) in wedges.

In section 3.2 we summerize some of the notions from [EG] and in section 3.3 we state and prove our main results. Also, throughout this chapter, $WF(u)$ will denote the C^∞ wave-front set of u.

3.2 Preliminaries

In this section we will briefly recall some of the notions and results we will need about involutive structures. The reader is refered to [EG] for more details. We assume

$(M, \mathcal{V})$ is an involutive structure and the fiber dimension of $\mathcal{V}$ equals n. A distribution f on M is called a solution if $Lf = 0$ for all smooth sections L of $\mathcal{V}$. A real cotangent vector $\sigma \in T_p^*M$ is said to be characteristic for the involutive structure $(M, \mathcal{V})$ if $\sigma(L) = 0$ for all $L \in \mathcal{V}_p$ and we let

$$T_p^0 = \{\sigma \in T_p^*M : \sigma \text{ is characteristic for } (M, \mathcal{V})\}.$$

Even when $\mathcal{V}$ is a line bundle, the dimension of T_p^0 may not be constant as p varies. However, when $\mathcal{V}$ is a CR structure, that is, $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$, then T^0 is a vector bundle.

Definition 52 *A smooth submanifold X of M is called maximally real if $\mathbb{C}T_pM = \mathcal{V}_p \oplus \mathbb{C}T_pX$ for each $p \in X$.*

If X is a maximally real submanifold and $p \in X$, define

$$\mathcal{V}_p^X = \{L \in \mathcal{V}_p : \Re L \in T_pX\}.$$

We recall the following result from [EG] which is also valid for a general involutive structure.

Proposition 53 *(Lemma II.1 in [EG]) $\mathcal{V}^X$ is a real subbundle of $\mathcal{V}|_X$ of rank n. The map*

$$\Im : \mathcal{V}|_X \to TM$$

which takes the imaginary part induces an isomorphism

$$\mathcal{V}^X \cong TM|_X \diagup TX.$$

Proposition 54 shows that when X is maximally real, for $p \in X$, $\Im$ defines an isomorphism from $\mathcal{V}_p^X$ to an $n-$dimensional subspace N_p of T_pM which is a canonical complement to T_pX in the sense that

$$T_pM = T_pX \oplus N_p.$$

Definition 54 *Let E be a submanifold of M, $\dim_{\mathbb{R}} E = k$. We say an open set $\mathcal{W}$ is a wedge in M at $p \in E$ with edge E if the following holds: there exists a diffeomorphism F of a neighborhood V of 0 in $\mathbb{R}^N$ $(N = \dim_{\mathbb{R}} M)$ onto a neighborhood U of p in M with $F(0) = p$ and a set $B \times \Gamma \subseteq V$ with B a ball centered at $0 \in \mathbb{R}^k$ an Γ a truncated, open convex cone in $\mathbb{R}^{N-k}$ with vertex at 0 such that*

$$F(B \times \Gamma) = \mathcal{W} \quad \textit{and} \quad F(B \times \{0\}) = E \cap U.$$

Definition 55 *Let E, $\mathcal{W}$ and $p \in E$ be as in the previous definition. The direction wedge $\Gamma_p(\mathcal{W}) \subseteq T_pM$ is defined as the interior of the set*

$$\{c'(0)|c : [0,1) \to M \text{ is } C^\infty,\ c(0) = p,\ c(t) \in \mathcal{W}\ \forall t > 0\}.$$

It is easy to see that $\Gamma_p(\mathcal{W})$ is a linear wedge in T_pM with edge T_pE. Set

$$\Gamma(\mathcal{W}) = \bigcup_{p \in E} \Gamma_p(\mathcal{W}).$$

Suppose $\mathcal{W}$ is a wedge in M with a maximally real edge X. As observed in [EG], since $\Gamma_p(\mathcal{W})$ is determined by its image in $T_pM \diagup T_pX$, the isomorphism $\Im$ can be used to define a corresponding wedge in $\mathcal{V}_p^X$ by setting

$$\Gamma_p^{\mathcal{V}}(\mathcal{W}) = \left\{L \in \mathcal{V}_p^X : \Im L \in \Gamma_p(\mathcal{W})\right\}.$$

$\Gamma_p^{\mathcal{V}}(\mathcal{W})$ is a linear wedge in $\mathcal{V}_p^X$ with edge $\{0\}$, that is, it is a cone. Define also

$$\Gamma_p^T(\mathcal{W}) = \left\{\Re L : L \in \Gamma_p^{\mathcal{V}}(\mathcal{W})\right\}.$$

$\Gamma_p^T(\mathcal{W})$ is an open cone in $(\Re\mathcal{V}_p) \cap T_pX$ (see [EG]). Set

$$\Gamma^{\mathcal{V}}(\mathcal{W}) = \bigcup_{p \in X} \Gamma_p^{\mathcal{V}}(\mathcal{W}) \ \text{ and } \ \Gamma^T(\mathcal{W}) = \bigcup_{p \in X} \Gamma_p^T(\mathcal{W}).$$

Definition 56 *Let $\mathcal{W}$ be a wedge in M with edge a maximally real submanifold X. We say a distribution $f \in \mathcal{D}'(\mathcal{W})$ is an approximate solution if $Lf \in L^1_{loc}(\mathcal{W})$ and*

$$Lf(p) = O(\mathit{dist}(p, X))^l \quad \forall l = 1, 2, 3, ...,$$

and for all smooth sections L of $\mathcal{V}$.

Definition 57 *Let $\mathcal{W}$ and X be as above, $f \in \mathcal{D}'(\mathcal{W})$ and $u \in \mathcal{D}'(X)$. Near a point $p \in X$ let $(x', x'') \in B \times \Gamma$ be a coordinate system where B and Γ are as in Definition 55. We say that f has a boundary value u if at each p and in each such coordinate system, f is a smooth function on Γ with values in $\mathcal{D}'(B)$, extends continuously to $\Gamma \cup \{0\}$ and equals u at $x'' = 0$.*

3.3 Edge of the Wedge Theory in Involutive Structures

Theorem 58 *Let $(M, \mathcal{V})$ be an involutive structure (not necessarily locally integrable), $\dim_{\mathbb{R}} M = m+n$, $rank_{\mathbb{C}}\mathcal{V} = n$, $X \subset M$ a maximally real submanifold, and $\mathcal{W}$ a wedge in M with edge X. Suppose that $u \in \mathcal{D}'(X)$ is the boundary value of an approximate solution $f \in \mathcal{D}'(\mathcal{W})$ of $\mathcal{V}f = 0$. Then*

$$WF(u) \subset \left(\Gamma^T(\mathcal{W})\right)^0.$$

Proof. Since $\mathcal{W}$ is a wedge in M with edge X, we get that near a point $p \in X$, (say, in $\Omega \subset M$), there are coordinates $(x,t) = (x_1, ..., x_m, t_1, ..., t_n)$ vanishing at p so that in Ω

$$X = \{(x,0) : |x| < r\} = B_r(0),$$

$$\mathcal{W} = X \times \Gamma \text{ for some open convex cone } \Gamma \subset \mathbb{R}^n_t.$$

Since X is maximally real,

$$\mathbb{C}TM = \mathbb{C}TX \oplus \mathcal{V}$$

and so for each $j = 1, \ldots, n$, there exists a smooth section L_j of $\mathcal{V}$ (near 0) and smooth functions $a_{jk}(x,t)$, $1 \le j \le n$, $1 \le k \le m$ such that

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk}(x,t)\frac{\partial}{\partial x_k} \quad (1 \le j \le n). \tag{3.1}$$

Observe that the L_j's are linearly independent over $\mathbb{C}$, and so

$$\mathcal{V} = \mathrm{span}_{\mathbb{C}}\{L_j : 1 \le j \le n\}.$$

Let

$$\{Z_1(x,t), ..., Z_m(x,t)\} \tag{3.2}$$

be smooth functions satisfying the following properties: for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|L_j Z_l(x,t)| \le C_N |t|^N, \text{ and } Z_l(x,0) = x_l, \text{ for } 1 \le l \le m. \tag{3.3}$$

For $l = 1, ..., m$, and $(x,t) \in \Omega$, we can write

$$Z_l(x,t) = x_l + \sum_{s=1}^{n} t_s \psi_{ls}(x,t), \tag{3.4}$$

where $\psi_{ls}(x,t) = \psi_{ls}^{(1)}(x,t) + i\psi_{ls}^{(2)}(x,t)$. Set

$$\begin{aligned} Z(x,t) &= (Z_1(x,t), ..., Z_m(x,t)), \text{ and} \\ A(x,t) &= \left(\psi_{ij}(x,t)\right)_{1\le i\le m,\ 1\le j\le n}. \end{aligned} \tag{3.5}$$

Then we can rewrite (3.4) in the matrix form

$$Z(x,t) = x + A(x,t)t. \tag{3.6}$$

Using (3.3), for all $1 \le j \le n,\ 1 \le l \le m$

$$-a_{jl}(0,0) = \psi_{lj}(0,0). \tag{3.7}$$

Hence, for all $1 \le j \le n,\ 1 \le l \le m$

$$-\Im a_{jl}(0,0) = \psi_{lj}^{(2)}(0,0). \tag{3.8}$$

We have:

$$\mathcal{V}_0^X = \{L \in \mathcal{V}_0 : \Re L \in T_0 X\} = \operatorname{span}_{\mathbb{R}}\{iL_j|_0 : 1 \le j \le n\}. \tag{3.9}$$

Indeed, the above span is contained in $\mathcal{V}_0^X$ and since its dimension over $\mathbb{R}$ is n, by Proposition 54, it equals $\mathcal{V}_0^X$. The direction wedge

$$\Gamma_0(\mathcal{W}) = \left\{\sum_{j=1}^m a_j \frac{\partial}{\partial x_j}|_0 + \sum_{j=1}^n b_j \frac{\partial}{\partial t_j}|_0 : a \in \mathbb{R}^m, b \in \Gamma\right\} \simeq \mathbb{R}^m \times \Gamma. \tag{3.10}$$

Hence,

$$\Gamma_0^{\mathcal{V}}(\mathcal{W}) = \left\{L \in \mathcal{V}_0^X : \Im L \in \Gamma_0(\mathcal{W})\right\} = \left\{\sum_{j=1}^n i b_j L_j|_0 : b \in \Gamma\right\}, \tag{3.11}$$

and

$$\begin{aligned} \Gamma_0^T(\mathcal{W}) &= \left\{\Re L : L \in \Gamma_0^{\mathcal{V}}(\mathcal{W})\right\} \\ &= \left\{\sum_{j=1}^n b_j \left(\sum_{k=1}^m -\Im A_{jk}(0,0)\frac{\partial}{\partial x_k}|_0\right) : b \in \Gamma\right\} \\ &= \left\{\sum_{j=1}^n b_j \left(\sum_{k=1}^m \psi_{kj}^{(2)}(0,0)\frac{\partial}{\partial x_k}|_0\right) : b \in \Gamma\right\} \\ &= \left\{\sum_{k=1}^m \left(\sum_{j=1}^n b_j \psi_{kj}^{(2)}(0,0)\right)\frac{\partial}{\partial x_k}|_0 : b \in \Gamma\right\} \subset T_0 X. \end{aligned} \tag{3.12}$$

Hence,

$$\begin{aligned}\left(\Gamma_0^T(\mathcal{W})\right)^0 &= \left\{\xi \in T_0^*X\backslash\{0\} \simeq \mathbb{R}^m\backslash\{0\} : \xi \cdot v \geq 0 \ \text{ for all } v \in \Gamma_0^T(\mathcal{W})\right\} \\ &= \left\{\xi \in \mathbb{R}^m\backslash\{0\} : \xi \cdot \Im A(0,0)b \geq 0 \ \text{ for all } b \in \Gamma\right\}. \end{aligned} \tag{3.13}$$

Therefore, since $\left(\Gamma_0^T(\mathcal{W})\right)^0$ is closed in $\mathbb{R}^m\backslash\{0\}$, we obtain

$$\xi^0 \notin \left(\Gamma_0^T(\mathcal{W})\right)^0 \Leftrightarrow \exists \text{ an open convex cone } \widetilde{\Gamma} \subset\subset \Gamma : \xi^0 \cdot \Im A(0,0)\widetilde{\Gamma} < 0. \tag{3.14}$$

For $j = 1, ..., n$, define the vector fields

$$L_j' = L_j - \sum_{k=1}^{m} L_j Z_k(x,t) M_k, \tag{3.15}$$

where $M_1, ..., M_m$ are C^∞ complex vector fields involving differentiation in the x variables only such that

$$M_k Z_l = \delta_{kl} \ \text{ for all } 1 \leq k \leq m, \ 1 \leq l \leq m. \tag{3.16}$$

Note that

$$L_j' Z_l = 0 \ \text{ for all } 1 \leq j \leq n, \ 1 \leq l \leq m. \tag{3.17}$$

If $g(x,t)$ is any C^1 function defined in Ω, observe that the differential

$$dg(x,t) = \sum_{j=1}^{n} L_j' g(x,t) dt_j + \sum_{k=1}^{m} M_k g(x,t) dZ_k. \tag{3.18}$$

Hence, if we consider the m-form

$$\omega(x,t) = g(x,t)\, dZ(x,t) = g(x,t)\, dZ_1 \wedge \cdots \wedge dZ_m(x,t), \tag{3.19}$$

its differential becomes

$$d\omega(x,t) = \sum_{j=1}^{n} L_j' g(x,t) dt_j \wedge dZ(x,t). \tag{3.20}$$

Since $f(x,t)$ is an approximate solution of $\mathcal{V}$ in $\mathcal{W}$,

$$\forall N \in \mathbb{N}\ \exists C_N > 0 : |L_j f(x,t)| \leq C_N\, |t|^N \ \text{ for all } (x,t) \in \mathcal{W}. \tag{3.21}$$

We also know that

$$\lim_{\Gamma \ni t \to 0} \int_X f(x,t)\varphi(x)\, dx = \langle u, \varphi \rangle \ \text{ exists for all } \varphi \in C_0^\infty(X).$$

Let $\eta(x) \in C_0^\infty(\mathbb{R}^m)$, $\eta(x) \equiv 1$ for $|x| \le r$, and $\eta(x) \equiv 0$ when $|x| \ge 2r$ (r small). We will consider the FBI transform of ηf:

$$\mathcal{F}_{\eta f}(t; y, \xi) = \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2}\eta(x)f(x,t)\left(\det Z_x(x,t)\right)\, dx. \tag{3.22}$$

where for $z \in \mathbb{C}^m$, we write $\langle z\rangle^2 = z_1^2 + \cdots + z_m^2$. Since the boundary value $bf = u$ exists, we have

$$\begin{aligned}\mathcal{F}_{\eta f}(0; y, \xi) &= \int_X e^{i\xi\cdot(y-x)-|\xi|\langle y-x\rangle^2}\eta(x)u(x)\, dx \\ &= \mathcal{F}_{\eta u}(y, \xi).\end{aligned} \tag{3.23}$$

Let $\xi^0 \in \mathbb{R}^m\backslash\{0\}$ be such that $\xi^0 \notin \left(\Gamma_0^T(\mathcal{W})\right)^0$. Then, by (3.14), we can get an open convex cone $\widetilde{\Gamma} \subset\subset \Gamma$ such that

$$\xi^0 \cdot \Im A(0,0)\widetilde{\Gamma} < 0. \tag{3.24}$$

Fix $T \in \widetilde{\Gamma}$ and let

$$\gamma(s) = sT \quad \text{for } 0 \le s \le 1.$$

Consider the m-form $\omega(x,t) = g(x,t)\, dZ(x,t)$, where

$$g(x,t) = e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2}\eta(x)f(x,t),$$

and it is to be understood that y and ξ are parameters. We now avail ourselves of Stokes' theorem

$$\int_\gamma \int_X d\omega(x,t) = \int_{\partial(X\times\gamma)} \omega(x,t). \tag{3.25}$$

Using (3.20), (3.25) becomes

$$\int_\gamma \int_X \sum_{j=1}^n L_j' g(x,t) dt_j \wedge dZ(x,t) = \int_X \omega(x,T) - \int_X \omega(x,0). \tag{3.26}$$

Note that by (3.17),

$$\begin{aligned}L_j' g(x,t) &= e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2}\eta(x)L_j' f(x,t) \\ &\quad + e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} f(x,t) L_j'\eta(x,t), \\ \omega(x,T) &= g(x,T)\left(\det Z_x(x,t)\right)\, dx \\ &= e^{i\xi\cdot(y-Z(x,T))-|\xi|\langle y-Z(x,T)\rangle^2}\eta(x)f(x,T)\left(\det Z_x(x,T)\right) dx, \text{ and} \\ \omega(x,0) &= g(x,0)\, dx = e^{i\xi\cdot(y-x)-|\xi|\langle y-x\rangle^2}\eta(x)u(x)\, dx.\end{aligned}$$

Hence, together with (3.26), the above equations imply

$$\begin{aligned} |\mathcal{F}_{\eta u}(y,\xi)| &\leq \left| \int_X e^{i\xi\cdot(y-Z(x,T))-|\xi|\langle y-Z(x,T)\rangle^2} \eta(x) f(x,T) \left(\det Z_x(x,T)\right)\, dx \right| \\ &+ \sum_{j=1}^{n} \left| \int_\gamma \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L'_j f(x,t) \left(\det Z_x(x,t)\right)\, dx dt_j \right| \\ &+ \sum_{j=1}^{n} \left| \int_\gamma \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} f(x,t) L'_j \eta(x) \det Z_x \, dx dt_j \right| \end{aligned} \quad (3.27)$$

Write

$$Q(x,t,y,\xi) = i\xi \cdot (y - Z(x,t)) - |\xi| \langle y - Z(x,t) \rangle^2 . \quad (3.28)$$

We have

$$\Re Q(x,t,y,\xi) = \xi\cdot\Im A(x,t)t - |\xi| \left[|y-x|^2 + |\Re A(x,t)t|^2 - |\Im A(x,t)t|^2 - 2\langle x-y, \Re A(x,t)t \rangle \right] . \quad (3.29)$$

Let $M > 0$ such that

$$\|A(x,t) - A(0,0)\| \leq M \left(|x| + |t|\right) \quad \text{for all } (x,t) \in \Omega$$

and so, for all $(x,t) \in \Omega$:

$$\xi \cdot \Im A(x,t)t \leq \xi \cdot \Im A(0,0)t + M\, |\xi|\, |t| \left(|x| + |t|\right) .$$

Therefore, for some $C > 0$,

$$\begin{aligned} \Re Q(x,t,y,\xi) &\leq \xi \cdot \Im A(0,0)t + M(|x| + |t|)|t||\xi| \\ &\quad + C|t|^2|\xi| - \frac{|y-x|^2}{2} |\xi| . \end{aligned}$$

Since $\xi^0 \cdot (\Im A(0,0)T) < 0$, there is a conic neighborhood $\mathcal{C}$ of ξ^0 and $c > 0$ such that

$$\xi \cdot (\Im A(0,0)t) \leq -2c|t||\xi| \quad \forall \xi \in \mathcal{C},\ \forall t \in \gamma.$$

Hence for r small enough, $|x| \leq r$, and $|t|$ small,

$$\Re Q(x,t,y,\xi) \leq -c|t||\xi| \quad \forall \xi \in \mathcal{C},\ \forall t \in \gamma.$$

Thus, there are $\delta > 0$, $C_0 > 0$, an open neighborhood $\mathcal{O} \subset \mathbb{R}^m$ of the origin and an open conic neighborhood $\mathcal{C} \subset \mathbb{R}^m \backslash \{0\}$ of ξ^0 such that for all $t \in \gamma$ and all $(y,\xi) \in \mathcal{O} \times \mathcal{C}$:

$$\Re Q(x,t,y,\xi) \leq -\frac{1}{4} C_0\, |t|\, |\xi| . \quad (3.30)$$

We are now ready to conclude the proof. Look back at (3.27). We have

$$\begin{aligned} & \left| \int_X e^{i\xi\cdot(y-Z(x,T))-|\xi|\langle y-Z(x,T)\rangle^2} \eta(x) f(x,T) \left(\det Z_x(x,T)\right) dx \right| \\ \leq & \int_X e^{-\frac{1}{4}C_0|T||\xi|} \left|\eta(x) f(x,T)\right| \left(\det Z_x(x,T)\right) dx \\ \leq & \; Ce^{-\frac{1}{4}C_0'|\xi|} \text{ for all } (y,\xi) \in \mathcal{O} \times \mathcal{C}. \end{aligned}$$

Since $L_j'\eta(x) \equiv 0$ for $|x| \leq r$, the term

$$\left| \int_\gamma \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} L_j'\eta(x) f(x,t) \left(\det Z_x(x,t)\right) dx dt_j \right|$$

has an exponential decay for y near 0 and ξ in a conic neighborhood of ξ_0. For N a positive integer,

$$\begin{aligned} & |\xi|^N \int_\gamma \left| \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j' f(x,t) dx \right| dt_j \\ \leq & \; C\left|\xi\right|^N \int_\gamma \left| \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j f(x,t) dx \right| dt_j \\ & + C|\xi|^N \sum_{k=1}^m \int_\gamma \left| \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j Z_k(x,t) M_k f(x,t) dx \right| dt_j. \end{aligned}$$

Since f is an approximate solution of the L_j's, we obtain

$$\begin{aligned} & C\left|\xi\right|^N \int_\gamma \left| \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j f(x,t) dx \right| dt_j \\ \leq & \; CC_N \int_\gamma \int_X e^{-\frac{1}{4}C_0|t||\xi|} \left|\xi\right|^N \left|t\right|^N dx dt_j \\ \leq & \; C' \text{ for all } (y,\xi) \in \mathcal{O} \times \mathcal{C}. \end{aligned}$$

Since $bf = u$ exists, so does $b\left(M_k f\right)$ for all $k = 1, ..., m$. Hence, after decreasing δ, we can find a positive integer n independent of N such that

$$\begin{aligned} & C\left|\xi\right|^N \sum_{k=1}^m \int_\gamma \left| \int_X e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j Z_k(x,t) M_k f(x,t) dx \right| dt_j \\ \leq & \; K_1 \left|\xi\right|^N \sum_{k=1}^m \int_\gamma \sup_{|\alpha|\leq n} \left| D_x^\alpha \left\{ e^{i\xi\cdot(y-Z(x,t))-|\xi|\langle y-Z(x,t)\rangle^2} \eta(x) L_j Z_k(x,t) \right\} \right| dt_j \\ \leq & \; K_2 e^{-\frac{1}{4}C_0|t||\xi|} \left|\xi\right|^N \left|t\right|^N \\ \leq & \; C'' \text{ for all } (y,\xi) \in \mathcal{O} \times \mathcal{C}. \end{aligned}$$

Therefore, for each $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that for all $(y, \xi) \in O \times \mathcal{C}$:

$$|\mathcal{F}_{\eta u}(y, \xi)| \leq \frac{C_N}{|\xi|^N}.$$

This shows that the FBI transform of u, $\mathcal{F}_u(x, \xi)$, has rapid decay in ξ for all $(x, \xi) \in O \times \mathcal{C}$. It is well known (e.g., see [BH3]) that this implies

$$\left(0, \xi^0\right) \notin WF_0(u).$$

This completes the proof. ■

We are now in a position to consider the Edge-of-the-Wedge Theorem:

Corollary 59 *(Edge-of-the-Wedge Theorem) Let $\mathcal{W}^+$ and $\mathcal{W}^-$ be wedges in Ω with edge X whose directions are opposite: $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$. If $u \in \mathcal{D}'(X)$ is the boundary value of an approximate solution f^+ of $\mathcal{V}$ on $\mathcal{W}^+$ and also the boundary value of an approximate solution f^- of $\mathcal{V}$ on $\mathcal{W}^-$, then $WF_p(u) \subset i_X^*(T_p^0\Omega)$.*

Proof. By the above theorem,

$$WF_p(u) \subset \left(\Gamma_p^T\left(\mathcal{W}^+\right)\right)^0 \cap \left(\Gamma_p^T\left(\mathcal{W}^-\right)\right)^0.$$

Note that

$$\Gamma_p^T\left(\mathcal{W}^+\right) = -\Gamma_p^T\left(\mathcal{W}^-\right).$$

Thus, if $\xi^0 \in WF_p(u)$, then

$$\xi^0 \cdot \Gamma_p^T\left(\mathcal{W}^+\right) \geq 0 \text{ and } \xi^0 \cdot \Gamma_p^T\left(\mathcal{W}^-\right) \geq 0.$$

This imples that

$$\xi^0 \cdot \Gamma_p^T\left(\mathcal{W}^+\right) = 0.$$

Since $\Gamma_p^T\left(\mathcal{W}^+\right)$ is open in $\Re\mathcal{V}_p \cap T_pX$, we conclude that

$$\xi^0 \in (\Re\mathcal{V}_p \cap T_pX)^\perp = i_X^*(T_p^0\Omega).$$

Thus, $WF_p(u) \subset i_X^*(T_p^0\Omega)$. ■

Corollary 60 *If $(M, \mathcal{V})$ is an elliptic structure and we have the same hypothesis as in the previous corollary, then u is C^∞ on X.*

There is a converse to Theorem 58:

Theorem 61 *Let $(M, \mathcal{V})$ be an involutive structure (not necessarily locally integrable),* $\dim_{\mathbb{R}} M = m+n$, *$rank_{\mathbb{C}}\mathcal{V} = n$, $X \subset M$ a maximally real submanifold, and $\mathcal{W}$ a wedge in M with edge X. Suppose $u \in \mathcal{E}'(X)$ is such that*

$$WF(u) \subset \left(\Gamma^T(\mathcal{W})\right)^0.$$

Then in a slightly smaller wedge $\mathcal{W}' \subset\subset \mathcal{W}$ with edge X, there exists an approximate solution $f \in \mathcal{D}'(\mathcal{W}')$ of $\mathcal{V}f = 0$ such that

$$u = bf \qquad \text{on } X.$$

Proof. We proceed exactly as in the proof of Theorem 58 until (3.13). For some open convex cone $\Gamma' \subset\subset \Gamma$, one can write

$$\mathcal{W}' = B_r(0) \times \Gamma'.$$

Using (3.13) and the fact that $\Gamma' \subset\subset \Gamma$, one can find an open convex cone $\mathcal{C} \subset \mathbb{R}^m \backslash \{0\}$ containing $\left(\Gamma_0^T(\mathcal{W})\right)^0$ and a constant $c > 0$ such that

$$\xi \cdot \Im A(0,0)t \geq c\,|\xi|\,|t| \qquad \text{for all } (\xi, t) \in \mathcal{C} \times \Gamma'. \tag{3.31}$$

For $(x,t) \in \mathcal{W}'$ and $\xi \in \mathcal{C}$, define

$$\begin{aligned} Q(x,t,\xi) &= i\xi \cdot Z(x,t) \\ &= i\xi \cdot (x + \Re A(x,t)t) - \xi \cdot \Im A(x,t)t. \end{aligned}$$

Using (3.31) and the fact that $\Im A(x,t)$ is of class C^1 near $(0,0)$, one obtains for some $M > 0$ and for all $(x,t) \in \mathcal{W}'$ and $\xi \in \mathcal{C}$:

$$\begin{aligned} \Re Q(x,t,\xi) &= -\xi \cdot \Im A(x,t)t \\ &\leq -\xi \cdot \Im A(0,0)t + M\,|\xi|\,|t|\ \ (|x| + |t|\ ) \\ &\leq -c\,|\xi|\,|t| + M\,|\xi|\,|t|\ \ (|x| + |t|) \end{aligned}$$

Choosing $0 < r, \delta < \frac{c}{4M}$, we can insure that

$$\Re Q(x,t,\xi) \leq -\frac{c}{2}\,|\xi|\,|t| \qquad \text{for all } (x,t,\xi) \in B_r(0) \times \Gamma'_\delta \times \mathcal{C}. \tag{3.32}$$

Since $u \in \mathcal{E}'(X)$, the Paley-Wiener theorem implies that there exists a constant $C > 0$ and a positive integer N such that the Fourier transform

$$|\widehat{u}(\xi)| \leq C(1 + |\xi|)^N \qquad \text{for all } \xi \in \mathbb{R}^m. \tag{3.33}$$

This allows us to define for $(x, t) \in B_r(0) \times \Gamma'_\delta$ the continuous function

$$\begin{aligned} f_1(x,t) &= \frac{1}{(2\pi)^m} \int\limits_{\mathcal{C}} e^{Q(x,t,\xi)} \widehat{u}(\xi) d\xi \\ &= \frac{1}{(2\pi)^m} \int\limits_{\mathcal{C}} e^{i\xi \cdot Z(x,t)} \widehat{u}(\xi) d\xi. \end{aligned} \tag{3.34}$$

We claim that

(i) f_1 is an approximate solution of $\mathcal{V}$;

(ii) $\displaystyle\iint\limits_{B_r(0)\times\Gamma'_\delta} |f_1(x,t)|\,|t|^N \,dxdt < \infty$ (N is the same as the one in (3.33)).

Assuming that the claims are true for the moment, we can use Theorem (50) to guarantee the existence of the boundary value of f_1, $bf_1 = \lim\limits_{\Gamma'_\delta \ni t \to 0} f_1(.,t)$, in $\mathcal{D}'(B_r(0))$ and we can use the formula obtained in that theorem to show that in fact

$$bf_1(x) = \frac{1}{(2\pi)^m} \int\limits_{\mathcal{C}} e^{i\xi \cdot x} \widehat{u}(\xi) d\xi. \tag{3.35}$$

Now, we show the validity of claims (i) and (ii) above. To show (i), we fix $t_0 \in \Gamma'_\delta$ and we consider a small open neighborhood of t_0 in Γ'_δ. In this small neighborhood, the dominated convergence theorem together with the estimate (3.32) allow us to pass L_j under the integral sign

$$L_j f_1(x,t) = \frac{1}{(2\pi)^m} \int\limits_{\mathcal{C}} i\xi \cdot L_j Z(x,t) e^{i\xi \cdot Z(x,t)} \widehat{u}(\xi) d\xi.$$

Since $Z(x, t)$ are approximate first integrals for $\mathcal{V}$, we get that for each $l = 1, 2, ...$ there exists a constant $C_l > 0$ such that

$$|L_j Z(x,t)| \leq C_l \,|t|^l \qquad \text{for all } (x,t) \in B_r(0) \times B_\delta(0). \tag{3.36}$$

There is a constant $K = K(c) > 0$ such that

$$|t|^N \,|\xi|^N \,e^{-\frac{c}{2}|\xi||t|} \leq K \qquad \text{for all } t \text{ and } \xi. \tag{3.37}$$

This implies, together with (3.36), that for each $l = 1, 2, ...$ there exists a constant $K_l > 0$ such that

$$|L_j f_1(x,t)| \leq K_l \, |t|^l \quad \text{for all } (x,t) \in B_r(0) \times \Gamma'_\delta. \tag{3.38}$$

Hence, f_1 is an approximate solution of $\mathcal{V}$ and claim (i) is proved. To prove claim (ii), we observe (using (3.37)) that there is a constant $C' > 0$ such that

$$|f_1(x,t)| \, |t|^N \leq C' \quad \text{for all } (x,t) \in B_r(0) \times \Gamma'_\delta.$$

Hence,

$$\iint\limits_{B_r(0)\times\Gamma'_\delta} |f_1(x,t)| \, |t|^N \, dxdt < \infty,$$

and claim (ii) follows. Now, for $x \in B_r(0)$, define

$$v(x) = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{R}^m \setminus \mathcal{C}} e^{i\xi \cdot x} \widehat{u}(\xi) d\xi. \tag{3.39}$$

Using the fact that $WF_0(u) \subset \left(\Gamma_0^T(\mathcal{W})\right)^0$, compactness of $(\mathbb{R}^m \setminus \mathcal{C}) \cap \mathbb{S}^{m-1}$, and the characterization of the C^∞ wavefront set by the rapid decay of the Fourier transform, we get that $v \in C^\infty(B_r(0))$. It is well known that in this case, one can find a C^∞ function $f_2 \in C^\infty(B_r(0) \times B_\delta(0))$ such that f_2 is an approximate solution of $\mathcal{V}$ and $bf_2 = v$ on X. Thus, from the Fourier Inversion formula, (3.35) and (3.39) we get that

$$u = bf_1 + bf_2 = bf,$$

where $f = f_1 + f_2$ is an approximate solution of $\mathcal{V}$ in the wedge $\mathcal{W}'$. This completes the proof. ■

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