Real Analysis Ph.D. Qualifying Exam
Temple University
January 12, 2018

Part I. (Do 3 problems)

1. Let a_n and ε_n be sequences of real numbers satisfying $|a_{n+1} - a_n| \le \varepsilon_n$ for all n with $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Prove that a_n converges to some a and $|a - a_n| \le \sum_{k=n}^{\infty} \varepsilon_k$.

2. Given two sets $A, B \subset \mathbb{R}^n$ define $A + B = \{x + y, x \in A, y \in B\}$. Prove that

   (a) if A is open or B is open, then $A + B$ is open,

   (b) if A is compact and B is closed, then $A + B$ is closed.

   (c) in $\mathbb{R}^2$ take $A = \{(x, 0) : x \in \mathbb{R}\}$ and $B = \{(y, 1/y) : y > 0\}$, show A and B are both closed and $A + B$ is not.

3. Let $f_n(x) = n\,x\,e^{-n\,x^2}$ on $[0, +\infty)$. Prove that

   (a) f_n converges to zero pointwise in $[0, +\infty)$

   (b) f_n does not converge uniformly in $[0, +\infty)$

   (c) f_n converges in measure on $[0, +\infty)$

   (d) $\int_0^{\infty} f_n(x)\,dx = \dfrac{1}{2}$.

   HINT for (c): may use that $e^z \ge z^2/2$ for all $z \ge 0$.

4. Suppose $f_n \to f$ a.e. on $\mathbb{R}^n$, f_n measurable. Prove that for each $\epsilon > 0$ there exist a sequence of disjoint measurable sets E_j of finite measure such that $|\mathbb{R}^n \setminus \cup_{j=1}^{\infty} E_j| < \epsilon$ and $f_n \to f$ uniformly on each E_j.

Part II. (Do 2 problems)

1. Let $b > 0$, $f \in L^1(0, b)$ and let $g(x) = \displaystyle\int_x^b \frac{f(t)}{t}\,dt$ for $0 < x < b$. Prove that $g \in L^1(0, b)$ and

$$\int_0^b g(x)\,dx = \int_0^b f(t)\,dt.$$

2. Let $1 \le p < \infty$, $f_k, f \in L^p(E)$, and let $a_k = \int_E |f_k(x) - f(x)|^p\,dx$. Suppose that $\sum_{k=1}^{\infty} a_k < \infty$. Prove that $f_k \to f$ a.e. in E as $k \to \infty$.

3. Let f be a continuous function in $[-1, 2]$. For $0 \le x \le 1$ and $k \ge 1$ let

$$f_k(x) = \frac{k}{2} \int_{x-\frac{1}{k}}^{x+\frac{1}{k}} f(t)\,dt.$$

   Prove that f_k is continuous in $[0, 1]$ and $f_k \to f$ uniformly in $[0, 1]$.