Mathematics Real Analysis Ph.D. Qualifying Exam
Temple University
January 12, 2007

December 20, 2006

All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable and all integrals are against Lebesgue measure. You may not use or refer to the Riemann integral in any of your answers; everything must be justified within the context of the Lebesgue theorems (MCT, DCT, LDT, ...).

Part I. (Select 3 questions.)

1. We say $f : \mathbf{R} \to \mathbf{R}$ is *superlinear* if
$$\lim_{x \to \pm\infty} \frac{f(x)}{|x|} = +\infty.$$
Show that f superlinear and differentiable implies $f'(\mathbf{R}) = \mathbf{R}$.

2. Given $a_0 > b_0 > 0$, let
$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n \geq 0.$$
Show that (a_n) is decreasing, (b_n) is increasing, and both sequences converge to the same limit.

3. Use the geometric series to show that
$$\sum_{n=1}^{\infty} \frac{n^k}{2^n}$$
is an integer for $k = 1, 2, 3, \ldots$.

4. Let $C \subset [0, 1]$ be the set of reals whose decimal expansion digits are zero or odd. Show that
$$C + C = \{x + y : x, y \in C\} = [0, 2].$$
(If $x + y = z$, look at the decimal expansions of x, y, z as geometric series in powers of $1/10$.)

Part II. (Select 2 questions.)

1. Define the Lebesgue measure $|A|$ of a set $A \subset \mathbf{R}^d$. Show that, if $|A| > 0$ and $\epsilon > 0$, there is a product of intervals $Q = I_1 \times I_2 \times \ldots \times I_d$ satisfying
$$|Q \cap A| > (1 - \epsilon)|Q|.$$

2. Let $f(x) = x^2 - 2$. By considering the minimum of $n^2|f(m/n)|$ over all naturals $n, m \geq 1$, show that
$$\left|\sqrt{2} - \frac{m}{n}\right| \geq \frac{1}{(2\sqrt{2} + 1)n^2}, \qquad n, m \geq 1.$$

3.