PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 2001

Part I. Do three (3) of these problems.

I.1. Let (a_n) and (ε_n) be sequences of positive numbers. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$, and there exists $0 \le \beta < 1$ such that

$$a_{n+1} \le \beta\, a_n + \varepsilon_n, \qquad \text{for all } n.$$

Prove that $\lim_{n\to\infty} a_n = 0$.

I.2. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of functions in $C[a,b]$. Prove that the function

$$g(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

is continuous in $[a,b]$.

I.3. Let $\phi \in L^1(\mathbb{R})$, $\phi \ge 0$, and $f(x,t) = e^{-|x|}\,\chi_{[-\phi(t),\phi(t)]}(x)$. Prove that $f \in L^1(\mathbb{R}^2)$. Here χ_E denotes the characteristic function of the set E.

I.4. Let $E \subset \mathbb{R}^n$ with $|E| < \infty$, $f_k \in L^1(E)$ such that $f_k \to 0$ in measure, and $\|f_k\|_{L^1(E)} \le M$ for all k. Prove that

$$\int_E |f_k\, g|^{1/2}\, dx \to 0, \qquad \text{as } k \to \infty,$$

for all $g \in L^1(E)$.

Part II. Do two (2) of these problems.

II.1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions and $a \in \mathbb{R}$. Suppose that

$$\sum_{n=1}^{\infty} |\{x \in \mathbb{R} : f_n(x) > a\}| < \infty.$$

Prove that $\limsup_{n\to\infty} f_n(x) \le a$, for a.e. $x \in \mathbb{R}$.

II.2. Let $K \subset \mathbb{R}^n$ be compact and $\{B_k\}_{k=1}^{\infty}$ a sequence of open balls whose union covers K. Prove that there exists a number $\varepsilon > 0$ such that given $x \in K$, the ball with center x and radius ε is contained in some B_k.

HINT: for each $x \in K$ pick a ball $B(x, r(x))$ such that the ball $B(x, 2r(x))$ is contained in some B_k. Select a finite subcovering of $B(x, r(x))$ and take the minimum of the corresponding $r(x)$.

II.3. An ellipsoid in $\mathbb{R}^n$ centered at the point x_0 is a set of the form

$$E = \{x \in \mathbb{R}^n : \langle A(x - x_0), x - x_0 \rangle \le 1\},$$

where A is an $n \times n$ positive definite and symmetric matrix, and $\langle , \rangle$ denotes the Euclidean inner product. Use the formula of change of variables for multiple integrals to show that the volume of E equals

$$\frac{\omega_n}{\sqrt{\det A}},$$

with ω_n the volume of the unit ball.