PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

Spring 1999

Justify carefully all reasoning.

Part I. Do three (3) of these problems.

I.1. Define the Lebesgue measure $|A|$ of a set $A \subset \mathbf{R}$. Show that

$$|A| = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n, A_n \text{ arbitrary} \right\}.$$

Here $\operatorname{diam}(A) = \sup\{|x-y| : x, y \in A\}$ is the diameter of A.

I.2. Show that

$$F(x) = \int_0^\infty \frac{\sin\left(xt^2\right)}{1+t^2}\, dt, \qquad x \in \mathbf{R}$$

is continuous, where "dt" denotes Lebesgue measure on $\mathbf{R}$.

I.3. Let f_n be a sequence of absolutely convergent continuous functions in $[a,b]$ such that $f_n(a) = 0$. Suppose that f_n' is a Cauchy sequence in $L^1[a,b]$. Show that there exists f, absolutely continuous in $[a,b]$, such that $f_n \to f$ uniformly in $[a,b]$.

I.4. Let f be a non-negative function on $\mathbf{R}$, let $g(x,y) = f(4x)f(x-3y)$, and let μ_n denote Lebesgue measure on $\mathbf{R}^n$. Suppose that $\int_{\mathbf{R}^2} g\, d\mu_2 = 2$. Calculate $\int_{\mathbf{R}} f\, d\mu_1$.

Part II. Do two (2) of these problems.

II.1. Let $f : [0,1] \to \mathbf{R}$ satisfy $1 \le f(x) \le 2$ and let

$$N(p) = \left(\int_0^1 f(x)^p\, dx \right)^{1/p}, \qquad p \neq 0.$$

(1) Compute $\lim_{p\to\infty} N(p)$.
(2) Compute $\lim_{p\to 0} N(p)$.
(3) Compute $\lim_{p\to -\infty} N(p)$.

II.2. Use the DCT on $(0,\infty)$ to compute

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n e^{it}\, dt. \qquad (i = \sqrt{-1})$$

II.3. Let E be the set of $x \in [0, 2\pi]$ such that $\lim_{n\to\infty} e^{inx}$ exists. Show that the Lebesgue measure of E is zero.