Part I. (Do 3 problems)

1. Let \( x_k \) be a sequence in a metric space \((X, d)\) such that \( \sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \infty \). Prove that \( x_k \) is a Cauchy sequence.

2. Prove that the function
   \[
   F(x) = \int_{0}^{+\infty} \frac{\cos(x t^2)}{1 + t^2} dt
   \]
   is well defined and is continuous for all \( x \in \mathbb{R} \).

3. Let \( E \subset \mathbb{R}^n \). The function \( f : E \to \mathbb{R} \) is upper semicontinuous at \( x_0 \in E \) if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( f(x) \leq f(x_0) + \epsilon \) for all \( |x - x_0| < \delta, x \in E \).
   Prove that if \( f \) is upper semicontinuous in \( E \) compact, then \( f \) is bounded above in \( E \).

4. Prove Dini’s theorem: Let \( X \) be a compact topological space. If \( f_n : X \to \mathbb{R} \) is a sequence of continuous functions such that \( f_n(x) \to 0 \) for each \( x \in X \) and \( f_n(x) \geq f_{n+1}(x) \) for all \( x \) and \( n \), then \( f_n \to 0 \) uniformly in \( X \).
   HINT: for \( \epsilon > 0 \) consider \( F_n = \{ x \in X : f_n(x) < \epsilon \} \).

Part II. (Do 2 problems)

1. Let \( f \in L^1(E) \). Prove that for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( A, B \subset E \) measurable with \( |A \triangle B| < \delta \) we have
   \[
   \left| \int_{A} f(x) \, dx - \int_{B} f(x) \, dx \right| < \epsilon.
   \]

2. Suppose \( f_k \to f \) a.e. on \( \mathbb{R}^n \), \( f_k \) measurable. Prove that for each \( \epsilon > 0 \) there exist a sequence of disjoint measurable sets \( E_j \) of finite measure such that \( |\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} E_j| < \epsilon \) and \( f_k \to f \) uniformly on each \( E_j \).

3. Let \( 0 < p \leq q < \infty \), \( f \in L^q(X, \mu) \), and \( E \subset X \) with \( 0 < \mu(E) < \infty \). Prove that
   \[
   \left( \frac{1}{\mu(E)} \int_{E} |f(x)|^p \, d\mu(x) \right)^{1/p} \leq \left( \frac{1}{\mu(E)} \int_{E} |f(x)|^q \, d\mu(x) \right)^{1/q}.
   \]