Mathematics Real Analysis Ph.D. Qualifying Exam — August 23, 2004

All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable, all integrals are against Lebesgue measure, and $L^p = L^p(\mathbf{R}^d)$. Throughout $\sum$ means $\sum_{n=1}^{\infty}$.

Part I. (Select 3 questions.)

1. Given $f_n : \mathbf{R}^d \to \mathbf{R}$, $n \geq 1$, assume that $\sum f_n$ converges uniformly on $\mathbf{R}^d$, and let $a \in \mathbf{R}^d$. Suppose that $\lim_{x\to a} f_n(x)$ exists for all $n \geq 1$. Show that $\lim_{x\to a} \sum f_n(x) = \sum \lim_{x\to a} f_n(x)$.
2. Show that a compact metric space has a countable dense subset.
3. Given an infinite sequence (c_n) of complex numbers, let $f(x) = \sum c_n e^{inx}$. Show:
    1. If $\sum |c_n| < \infty$, then f is continuous on $\mathbf{R}$.
    2. If $\sum n|c_n| < \infty$, then f is continuously differentiable on $\mathbf{R}$.
4. Find all p, q real such that the integral $\int_0^1 x^p(-\log x)^q dx$ is finite.

Part II. (Select 2 questions.)

1. Suppose that f is continuous on $[0, 1]$, differentiable on $(0, 1)$, $f(0) = 0$, and $f'(0+)$ exists. Show that $f(x)x^{-3/2}$ is integrable over $(0, 1)$.
2. Let $g : \mathbf{R}^d \to \mathbf{R}$ be nonnegative with $\int_{\mathbf{R}^d} g(y)\, dy = 1$, let $g_\epsilon(x) = \epsilon^{-d} g(x/\epsilon)$ for $\epsilon > 0$, and define $f_\epsilon(x) = (g_\epsilon \star f)(x) = \int_{\mathbf{R}^d} g_\epsilon(y) f(x - y)\, dy$. If $f : \mathbf{R}^d \to \mathbf{R}$ is continuous and bounded, then show that $f_\epsilon \to f$ as $\epsilon \to 0$ uniformly on compact subsets of $\mathbf{R}^d$ (Hint — change of variables).
3. Fix $p \geq 1$. Assume without proof that the set of continuous functions with compact support is dense in L^p. Let $f \in L^p$ and, for $t \in \mathbf{R}^d$, let $f_t(x) = f(x + t)$ be the translate of f by t. Show
    1. The map $t \mapsto f_t$ is continuous from $\mathbf{R}^d \to L^p$, and
    2. $f_\epsilon \to f$ in L^p as $\epsilon \to 0$, where $f_\epsilon = g_\epsilon \star f$.