PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

Fall 1995

Part I. Do three (3) of these problems.

I.1. Let μ be the Lebesgue measure on $\mathbb{R}$. Let $\phi(x) = x^2$. Define a measure ν by

$$\nu(A) = \mu(\phi^{-1}(A)), \text{ for all Lebesgue measurable sets } A.$$

Find the Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ if they exist.

I.2. Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. Show:

(1) $\Gamma(x) < \infty$ for all $x > 0$;
(2) $\Gamma'(x) = \int_0^\infty e^{-t} t^{x-1} \ln t \, dt$ if $x > 0$.

I.3. Let $A \subset \mathbb{R}$ be a measurable set with positive measure. Show there is an interval I such that the measure of $I \cap A$ is larger than 99% percent of the measure of I.

I.4. (1) Give an example of a sequence of functions f_n defined on $\mathbb{R}$ such that, as $n \to \infty$, $f_n \to 0$ in measure, but f_n does not converge to 0 almost everywhere.

(2) Give an example of a sequence of functions $f_n \in L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g \, dx = 0 \text{ for all } g \in L^2(\mathbb{R}),$$

but f_n does not converge to 0 as $n \to \infty$, in $L^2(\mathbb{R})$.

Part II. Do two (2) of these problems.

II.1. Let $f \in L^p([0,1], dx)$, $1 < p < \infty$. Let $F(x) = \int_0^x f(t) \, dt$. Show that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h^{1-\frac{1}{p}}} = 0.$$

II.2. Let H be a Hilbert space with inner product (,). Let $\{e_\alpha\}_{\alpha \in I}$ be an orthonormal basis for H. Consider a sequence of elements $\{x_n\}$ in H. Show that

$$\lim_{n\to\infty} x_n = x \text{ in the weak topology}$$

if and only if

i) $\lim_{n\to\infty}(x_n, e_\alpha) = (x, e_\alpha)$ for all $\alpha \in I$;
ii) $\sup_n \|x_n\| < \infty$

II.3. Given a Lebesgue-integrable function f on $\mathbb{R}$, set $F(a) = \int_{-\infty}^\infty f(x) \cos(ax) \, dx$. Show that F is uniformly continuous at every point.

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