PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

Fall 1994

Part I. Do three (3) of these problems.

I.1. (a) Give an example of a function $f(x)$ such that $\lim_{m\to\infty} \int_0^m f(x)\,dx$ exists, but $\lim_{m\to\infty} \int_0^m |f(x)|\,dx$ does not exist.

(b) Give an example of a function $f(x)$ such that $\lim_{\varepsilon\to 0} \int_0^\varepsilon f(x)\,dx$ exists, but $\lim_{\varepsilon\to 0} \int_0^m |f(x)|\,dx$ does not exist.

I.2. Give an example of a countable dense subset for each of the following:

(a) ℓ^2 (in the ℓ^2 norm).
(b) $L^2[0,1]$ (in the L^2 norm).
(c) $L^1[0,1]$ (in the L^1 norm).

I.3. Let f_n be a sequence of absolutely convergent continuous functions in $[a,b]$ such that $f_n(a) = 0$. Suppose that f_n' is a Cauchy sequence in $L^1[a,b]$. Show that there exists f, absolutely continuous in $[a,b]$, such that $f_n \to f$ uniformly in $[a,b]$.

I.4. Let f be a non-negative function in $\mathbb{R}$. Suppose that the double integral

$$\iint_{\mathbb{R}^2} f(4x)f(x-3y)dxdy = 2.$$

Calculate $\int_{-\infty}^{\infty} f(x)dx$.

Part II. Do two (2) of these problems.

II.1. Prove: Every L^1 function is continuous in the L^1 norm, that is,

$$\lim_{h\to 0} \int_0^1 |f(x+h) - f(x)|dx = 0$$

Note: You may assume f vanishes outside $[0,1]$.

II.2. Given a collection of closed subintervals of $[0,1]$ such that any two of the subintervals have a point in common, prove that all of them have a point in common.

II.3. Let $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $g \in L^q[0,1]$, then

$$\ell(f) = \int_0^1 f(x)g(x)/,dx$$

is a continuous linear functional on $L^p[0,1]$. Find $\|\ell\|$