PDEs Ph.D. Qualifying Exam
Temple University
January 7, 2016

Part I. (Do 3 problems)

1. Solve the Cauchy Problem

$$\begin{cases} x_1 \dfrac{\partial u}{\partial x_1} + 2\,x_2 \dfrac{\partial u}{\partial x_2} + \dfrac{\partial u}{\partial x_3} = 3\,u \\ u(x_1, x_2, 0) = f(x_1, x_2) \end{cases}$$

   where $f(x_1, x_2) \in C^1(\mathbb{R}^2)$.

2. A function f is radial if its value at x depends only on $|x|$. Prove that a radial harmonic function on the ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is constant. Is this true if we replace B by the punctured ball $\{x \in \mathbb{R}^n : 0 < |x| < 1\}$? Justify your answer.

3. Solve the Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = \cos x \\ u(x, 0) = \sin x, \\ u_t(x, 0) = x + 1. \end{cases}$$

4. Use the Fourier transform to prove that if $f \in L^1(\mathbb{R}^n)$ satisfies $f = f * f$, then $f \equiv 0$.

Part II. (Do 2 problems)

1. Suppose u solves the equation $\Delta u = 1$ in the unit ball $x^2 + y^2 \leq 1$ of $\mathbb{R}^2$ and satisfies $u(x, 0) = u_y(x, 0) = 0$ for $-1 < x < 1$. Find the polynomial $p(x, y)$ of degree two satisfying

$$u(x, y) = p(x, y) + o(|(x, y)|^3)$$

   as $(x, y) \to (0, 0)$.

2. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ solve the equation $\Delta u + V \cdot Du = F$ in Ω bounded domain, where V is a continuous vector field in Ω, $F > 0$ in Ω, and $u < 0$ on $\partial\Omega$. Prove that $u < 0$ in Ω.

3. Let $U \subset \mathbb{R}^n$ be open and $u \in W^{1,1}_{\text{loc}}(U)$. Show that $|u| \in W^{1,1}_{\text{loc}}(U)$