PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let $\alpha \in \mathbb{R}$.

   (a) If $f \in C^1(\mathbb{R})$, then show that the function $u(x, y) = f(x - \alpha y)$ is a solution of the first order pde
   $$\alpha u_x + u_y = 0.$$

   (b) If $f \in C(\mathbb{R})$, then show that $u(x, y) = f(x - \alpha y)$ is a weak solution, that is, for every $\phi \in C_0^1(\mathbb{R}^2)$
   $$\iint \Big(\alpha\phi_x(x, y) + \phi_y(x, y)\Big)\, u(x, y)\, dxdy = 0.$$

2. Let $f \in L^1(\mathbb{R}^n)$. Prove that the Fourier transform $\hat{f}$ is uniformly continuous in $\mathbb{R}^n$.

3. Let u be harmonic on a bounded domain Ω in $\mathbb{R}^n$, $u \in C(\Omega)$. Prove that
   $$|Du(x)| \leq \frac{C_n}{\text{dist}\,(x, \partial\Omega)} \max_{\Omega} |u|$$
   for all $x \in \Omega$, where C_n is a constant depending only on the dimension n.

   HINT: Either use the Poisson kernel for a ball or use the fact that derivatives of u are also harmonic, the mean value property and the divergence theorem.

4. Suppose u is a C^2 solution of the wave equation
   $$u_{tt} - u_{xx} = 0, \;\; x \in \mathbb{R}, \; t > 0, \;\; u(x, 0) = f(x), \;\; u_t(x, 0) = g(x).$$
   If $f(x) = 0 = g(x)$ on the interval $[-R, R]$, up to what time t can you guarantee that $u = 0$ at the center $x = 0$? Justify your answer.

Part II. (Do 2 problems)

1. Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in W^{1,2}(\Omega)$. Show that $v(x) = F(u(x)) \in W^{1,2}(\Omega)$ and the weak derivatives $v_{x_i} = F'(u)u_{x_i}$ for $1 \leq i \leq n$.

2. Let $u(x,t)$ be a C^2 bounded solution of

$$u_t(x,t) - u_{xx}(x,t) = 0,\ \ x \in \mathbb{R},\ t > 0,\ u(x,0) = f(x)$$

where $f \in C(\mathbb{R})$ satisfies:

$$\lim_{x \to +\infty} f(x) = A,\ \ \lim_{x \to -\infty} f(x) = B$$

for some constants A and B. Show that $\lim_{t \to \infty} u(x,t) = \dfrac{A+B}{2}$, for each $x \in \mathbb{R}$.

Hint: Use an integral representation for the solution u. Justify why the representation is valid.

3. Suppose $u \in C^2(\bar{\Omega})$ is harmonic and $\partial\Omega$ is smooth. Prove that

$$\int_{\partial\Omega} u \frac{\partial u}{\partial \nu}\, d\sigma(x) \geq 0,$$

with strict inequality unless u is constant.