PDEs Ph.D. Qualifying Exam
Temple University
August 21, 2014

Part I. (Do 3 problems)

1. Solve the initial value problem

$$u_x + x^2\, y\, u_y = -u, \qquad u(0, y) = y^2.$$

2. The Fourier transform is defined by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$. Calculate the Fourier transform of the function

$$f(x) = \begin{cases} \dfrac{e^{2\pi i b x}}{\sqrt{a}} & \text{for } |x| \le a \\ 0 & \text{for } |x| > a, \end{cases}$$

and the norm $\|\hat{f}\|_2$. The numbers a and b are positive.

3. Let Ω be a bounded open set in $\mathbb{R}^n$. Prove the following interpolation inequality:

$$\left(\int_\Omega |Du(x)|^2\, dx\right)^2 \le \left(\int_\Omega u(x)^2\, dx\right)\left(\int_\Omega (\Delta u(x))^2\, dx\right)$$

for all $u \in C_0^\infty(\Omega)$; Du denotes the gradient of u.

HINT: first prove the following formula valid for all $v \in C^\infty$: $\operatorname{div}(v\, Dv) = v\Delta v + |Dv|^2$.

4. Let u be a bounded solution to the heat equation $u_t - u_{xx} = 0$ in $-\infty < x < \infty, t > 0$ with $u(x, 0) = f(x)$ with $f \in L^2(\mathbb{R})$. Prove that there is a constant $C > 0$, independent of u, such that

$$\sup_x |u_x(x,t)| \le C\, t^{-3/4}\, \|f\|_2, \qquad \text{for all } t > 0.$$

Part II. (Do 2 problems)

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. Prove that if $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $1 \le p, q \le \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $uv \in W^{1,1}(\Omega)$.

2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and let u_k be a sequence of harmonic functions in Ω. Suppose that $u_k \le u_{k+1}$ for $k = 1, 2, \cdots$ and there exists $x_0 \in \Omega$ such that $u_k(x_0)$ converges. Prove that there exists a harmonic function u in Ω such that $u_k \to u$ uniformly on compact subsets of Ω.

3. Let Ω be a bounded smooth domain and let u be smooth in $\bar{\Omega} \times [0, T]$ solving

$$\begin{aligned} u_{tt} - \Delta u + u^3 &= 0 \quad \text{in } \Omega \times [0, T] \\ u(x,t) &= 0 \quad \text{in } \partial\Omega \times [0, T]. \end{aligned}$$

Prove that the energy

$$E(t) = \int_\Omega \left(u_t^2 + |Du|^2 + \frac{1}{2}u^4\right) dx$$

is constant in $[0, T]$.