Numerical Analysis Qualifying Written Exam (August 2019)

Part I: do 3 of 4

1. Use an appropriate numerical method using the appropriate choice of parameters to evaluate (the exact value of) the following integral

$$\int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

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2. Consider the data $f(-2) = 6$, $f(-1) = 2$, $f(1) = 3$, $f(2) = 10$, and assume that f is a smooth function.
(a) Find $p_3(x)$, the cubic interpolating polynomial. Explain the method and formulas you use to find it. Give and estimate of the error $p(x) - f(x)$ for x around $x = 0$.
(b) The data suggest that f has a minimum between $x = -1$ and $x = 1$. Find an approximate value for the location $x_{\min}$ of the minimum.
(c) If in addition you know that $f'(-2) = -1$, construct an interpolating polynomial $p(x)$ so that $p'(-2) = -1$. Explain the method and formulas you use to find it. Give and estimate of the error $p(x) - f(x)$ for x around $x = 0$.

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3. For the ODE $u'(t) = f(u(t), t)$, consider the numerical method

$$\begin{aligned} s_1 &= f(U^n, t_i) \\ s_2 &= f(U^n + k\, s_1, t_n + k) \\ U^{n+1} &= U^n + k\left(\tfrac{1}{2}s_1 + \tfrac{1}{2}s_2\right) \end{aligned}$$

(a) What type of method is it (Runge-Kutta, multistep, Taylor series, etc.)?
(b) Write the Butcher tableau for this scheme.
(c) Calculate the scheme's local truncation error.
(d) Prove (i.e. provide rigorous arguments, possibly referencing an important theorem or fact) why the scheme is consistent and why it is convergent.
(e) Provide a graphical interpretation (sketch) of what one step of this scheme does.

4. For approximating the solutions of ODE IVPs $u'(t) = f(u(t), t)$, $u(0) = \eta$:
(a) Name a key advantage of implicit methods vs. explicit methods. Name a key drawback. Name a type of problem where implicit methods should be used rather than explicit, and explain why.
(b) Name a key advantage of adaptive time stepping vs. fixed step-size methods. Name a key drawback. Name a type of problem where adaptive time steping methods should be used rather than a fixed step-size, and explain why.
(c) Name a key advantage of multistep methods vs. Runge-Kutta methods. Name a key drawback. Name a type of problem where multistep methods are a better choice than Runge-Kutta methods, and explain why.

Part II: do 2 of 3

1. Consider the function $f(x) = e^x \sin\left(\frac{x\pi}{4}\right)$, $x \in [0,4]$.
(a) Show that there is a *unique* fixed point x^* in this interval, i.e., $x^* \in [0,4]$ so that $x^* = f(x^*)$.
(b) Explain why the standard fixed point iteration $x_{k+1} = f(x_k)$ is not guaranteed to converge in this case. Give algebraic and geometric explanations if possible.
(c) Propose an appropriate numerical method to find this fixed point, provide the appropriate theory on why this method would converge for this function. Comment on the rate of convergence of the method proposed.

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2. Consider the following two time-stepping schemes:
(B) $U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}f(U^{n+2}, t_{n+2})$
(C) $U^{n+1} = U^n + k\frac{1}{2}\left(f(U^{n+1}, t_{n+1}) + f(U^n, t_n)\right)$.
(a) Prove that both schemes are zero-stable.
(b) Prove that both schemes are A-stable. [Hint: If the boundary locus method is used for (B), the fact that the curve $\frac{3}{2} - 2e^{i\theta} + \frac{1}{2}e^{2i\theta}$ does not penetrate the left half plane can be used.]
(c) Prove that one scheme is L-stable, while the other is not. [Hint: the solutions of $a\zeta^2 - b\zeta + c = 0$ approach 0 if $a \to \infty$ while b, c fixed.]
(d) Explain for what types of problems A-stability is important, and for what types of problems L-stability is important.

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4. For the ODE $u'(t) = f(u(t), t)$:
(a) Write the Crank-Nicolson method.
(b) Derive the equation for the region of absolute stability for this scheme.
(c) Draw the region of absolute stability in the complex plane.
(d) Explain what a stiff problem is.
(e) Discuss: For what types of stiff problems is the Crank-Nicolson scheme a good choice, factoring in absolute stability, as well as L-stability arguments.