Comprehensive Exam in Geometry & Topology

Temple University

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Part I: Do three of the following problems.

1. Prove that the equations

$$x^2 - y^2 - z^2 + w^2 - 3z = 0$$
$$2xy - 2zw - 3w = 0$$

define a submanifold of $\mathbb{R}^4$. Find its dimension and compute the tangent space at $(0,0,0,0)$.

2. Suppose that M, N are compact, connected, smooth, manifolds without boundary, and that they have equal dimension. Let $f\colon M \to N$ be an immersion. Show that f is a finite-sheeted covering map.

3. Let X be a cell complex such that $H_1(X) = \mathbb{Z}/3$. Let T^3 be the 3–torus. Prove that every continuous map $f\colon X \to T$ is homotopic to a constant map.

4. State and prove the Brouwer fixed point theorem in n dimensions. You may assume standard results about algebraic or smooth invariants of standard spaces.

Part II: Do two of the following problems.

1. Let $X = S^1 \vee S^1$ be the figure-8 graph with loops labeled a, b. Let $f\colon X \to X$ be a map such that $f_*(a) = ba$ and $f_*(b) = bab$. Let Y be the *mapping torus* of f:

$$Y = X \times [0,1]/\sim, \quad \text{where} \quad (x,0) \sim (f(x),1).$$

Construct a cell complex structure on Y, and use it to give a presentation of $\pi_1(Y)$.

2. Let M be a compact, boundaryless, simply connected 4–manifold such that $\chi(M) = 4$ and $H_2(M)$ is torsion–free. Let $K \subset M$ be a knot (that is, a smoothly embedded copy of S^1).

(a) Use Poincaré duality to compute $H_i(M)$ for every i.

(b) Let $N = M \smallsetminus K$. Use the Mayer–Vietoris sequence to compute $H_i(N)$ for every i.

3. The following questions are about $M = \mathbb{R}^3 \smallsetminus \{0\}$.

(a) Prove that there exists a form $\alpha \in \Omega^2(M)$ which is closed but not exact.

(b) Let S be an embedded 2–sphere in M, not necessarily centered at the origin of $\mathbb{R}^3$. Recall that S divides $\mathbb{R}^3$ into an inside and an outside. Show that $\int_S \alpha = 0$ if and only if the origin is on the outside. Here, α is the same 2–form as in part (a).