Ph.D. Comprehensive Examination
Complex Analysis
January 2021

Part I. Do three of these problems.

I.1. Compute using contour integration
\[ \int_0^{2\pi} \frac{\cos \theta}{\cos \theta - i} d\theta. \]

I.2. Let \( G = \{ z \in \mathbb{C} : |z| < 1 \} \). Let
\[ K = \left\{ z \in \mathbb{C} : \frac{1}{4} \leq |z| \leq \frac{3}{4} \right\}. \]
Show that there exists a function \( f \) analytic on some open set \( G_1 \) containing \( K \), which cannot be approximated by functions analytic in \( G \).

I.3. Give an example of an unbounded function, analytic in \( \mathbb{C} \setminus \{ z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \leq 0 \} \) (complex plane cut along the negative real line), such that
\[ \lim_{z \to x} |f(z)| \leq 1, \quad \forall x \leq 0. \]

I.4. Suppose that a sequence of analytic functions \( f_n : \mathcal{H}^+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \to \mathbb{C} \) satisfies \( \text{Im}(f_n(z)) > 0 \) for all \( z \in \mathcal{H}^+ \) and all \( n \geq 1 \). Suppose that for each \( z \in \mathcal{H}^+ \) the limit
\[ f(z) = \lim_{n \to \infty} f_n(z) \]
exesists. Prove that \( f(z) \) is analytic in \( \mathcal{H}^+ \) and that convergence is uniform on compact subsets of \( \mathcal{H}^+ \).

Part II. Do two of these problems.

II.1. Let \( G = \{ z \in \mathbb{C} : |z| > 1 \} \). Suppose \( f : G \to \mathbb{C} \) is analytic and there exists a sequence \( z_n \to \infty \), such that \( z_n^2 f'(z_n) \to 0 \). Prove that \( f \) cannot be injective on \( G \).

II.2. Let \( f \) be analytic in \( B^-(0,R) = \{ z \in \mathbb{C} : |z| \leq R \} \) with \( f(0) = 0 \), \( f'(0) \neq 0 \), and \( f(z) \neq 0 \) for \( 0 < |z| \leq R \). Put
\[ \rho = \min\{|f(z)| : |z| = R\} > 0. \]
Define \( g : B(0,\rho) \to \mathbb{C} \) by
\[ g(\omega) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{zf'(z)}{f(z) - \omega} \, dz. \]
Show that \( z = g(\omega) \) is the unique solution of \( f(z) = \omega, |z| < R \), provided \( \omega \in B(0,\rho) \).

Hint: Use argument principle to show uniqueness of solution and the residue theorem to show that the solution is \( g(\omega) \).

II.3. A Poincaré line is an arc of a circle orthogonal to the unit circle \( |z| = 1 \) that lies inside the unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Use conformal automorphisms of the unit disk \( U \) to write parametric equations of the Poincaré line passing through two given points \( \{ a, b \} \subset U \).