Ph.D. Comprehensive Examination
Complex Analysis

January 2017

Part I. Do three of these problems.

I.1 Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$ and consider $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$, a continuous function which is analytic in $\mathbb{D}$ and satisfies

$$|f(z)| = 1 \qquad \forall\, z \in \partial\mathbb{D}.$$

Show that if f has no zeros in $\mathbb{D}$ then f is constant on $\overline{\mathbb{D}}$. Here, as usually $\overline{\mathbb{D}}$ denotes the closure of the set $\mathbb{D}$ in $\mathbb{C}$, and $\partial\mathbb{D}$ denotes the topological boundary of $\mathbb{D}$.

I.2 Evaluate $\displaystyle\int_0^\infty \frac{1}{(x+3)\sqrt{x}}\, dx.$

I.3 Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a positive harmonic function. Show that u is constant.

I.4 Show that the family of all analytic maps $f : \mathbb{D} \longrightarrow \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ such that $|f(0)| \le 1$ is normal. That is, show that every sequence of functions in the family has a subsequence that converges uniformly on compact subsets. Here $\mathbb{D}$ is the open unit disc in $\mathbb{C}$.

Part II. Do two of these problems.

II.1 Find a conformal map from the open semi-disc $\mathbb{G} := \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Im}(z) > 0\}$ onto the half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, where as usually, Im denotes imaginary part.

II.2 Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc and assume that $f : \mathbb{D} \longrightarrow \mathbb{D}$ is an analytic function which is not the identity. Show that f can have, at most, one fixed point.

II.3 Let f be an entire function with the property that for each $w \in \mathbb{C}$, the equation $f(z) = w$ has precisely two solutions counted with multiplicity. Prove that f is a polynomial of degree two.

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.