Ph.D. Comprehensive Examination
Complex Analysis

January 2016

Part I. Do three of these problems.

I.1 Find a constant $C \in \mathbb{C}$ such that the function

$$\frac{1}{z^4 + z^3 + z^2 + 5z - 8} - \frac{C}{z - 1}$$

is analytic in a neighborhood of $z = 1$.

I.2 Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function and consider $u, v : \mathbb{C} \longrightarrow \mathbb{R}$ given by $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Show that if $u^2(z) \geq v^2(z)$ for each $z \in \mathbb{C}$ then f is a constant.

I.3 Consider the function $f : A \longrightarrow \mathbb{C}$ given by $f(z) = 1/z^3$ for each $z \in A$, where A is the annulus $A := \{z \in \mathbb{C} : 1 < |z| < 2\}$. Show that if $\{p_n\}_{n \in \mathbb{N}}$, $p_n : \mathbb{C} \longrightarrow \mathbb{C}$ for each $n \in \mathbb{N}$, is a sequence of polynomials, then

$$\{p_n\}_{n \in \mathbb{N}} \ \text{ does not converge uniformly to } f \text{ on } A \text{ as } n \to \infty.$$

I.4 Let $G \subset \mathbb{C}$ be open and assume that $f : G \to \mathbb{C}$ is a continuous function with the property that whenever γ is a continuous piecewise linear path in G, then

$$\int_\gamma f(z)\, dz$$

depends only on the endpoints of γ (and not on the specific path γ joining them). Show that f is holomorphic.

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1 Let G be a simply connected region other than $\mathbb{C}$, and let $a \in G$. Let $f : G \to G$ be holomorphic and such that $f(a) = a$. Show that $|f'(a)| \leq 1$, and, moreover, that if $|f'(a)| = 1$, then f is bijective.

II.2 Show that the series in

$$f(z) = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Big(\frac{1}{z-n} - \frac{1}{n}\Big)$$

converges uniformly on any compact set $K \subset \mathbb{C}$ disjoint from $\mathbb{Z}$, and that the resulting function satisfies $f(z+m) = f(z)$ for every $m \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

II.3 Let $G \subset \mathbb{C}$ be a bounded neighborhood of 0, $f : G \to G$ holomorphic and such that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = z$. Hint: Estimate the coefficients the Taylor expansion at 0 of all the iterated compositions $f^n = f \circ \cdots \circ f$ (n times) independently of n, then analyze what this says if $f(z) = z + z^m h(z)$ for some holomorphic h and $m > 1$. The required estimates depend critically on the boundedness of G.