Ph.D. Comprehensive Examination
Complex Analysis
August 2020

Part I. Do three of these problems.

I.1. Let $\ln(z)$ be the principal branch of the logarithm. Let G be the complex plane with rays $(-\infty, 0]$ and $[1, +\infty)$ removed. Prove, without any explicit construction, that there is an analytic branch of $\ln \ln(z)$ in G, i.e. there exists $f \in H(G)$ satifying $e^{f(z)} = \ln(z)$.

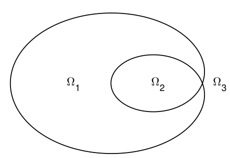
I.2. Let $f(z)$ be a principal branch of the square root. For every $|\alpha| < \pi$ find the radius of convergence r_α of the Taylor series of $f(z)$ centered at $a = e^{i\alpha}$.

I.3. Let $f : \mathcal{H}_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \to \mathbb{C}$ be analytic. Suppose $\operatorname{Im}(f(z)) \geq 0$ for all $z \in \mathcal{H}_+$. Prove that $f(z)$ has to be a constant function if there is $z_0 \in \mathcal{H}_+$, such that $\operatorname{Im}(f(z_0)) = 0$.

I.4. Find a Möbius transformation that maps half of the unit disk
$G = \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Im}(z) > 0\}$ into the wedge $W = \{z \in \mathbb{C} : |\operatorname{Arg}(z)| < \pi/4\}$.

Part II. Do two of these problems.

II.1. Suppose $f(z)$ is analytic in a region containg the closed unit disk. The curve in the figure below is the image of the unit circle under $f(z)$. How many preimages of the points $w \in \Omega_1$, $w \in \Omega_2$, and $w \in \Omega_3$ are there in the unit disk?



II.2. Let $P(z)$ be a polynomial of degree $n \geq 2$ that does not have any nonnegative real roots. Compute

$$\int_0^\infty \frac{dx}{P(x)}$$

using residues, assuming that $P(z)$ has only simple roots $\{z_1, \ldots, z_n\}$ in the complex plane. Hint: $f(z) = \ln(-z)/P(z)$ is meromorphic in the complex plane with nonnegative reals removed.

II.3. Let D be an open unit disk. Suppose $\{f_n : n \geq 1\}$ is a sequence of functions in $H(D)$ with Taylor series

$$f_n(z) = \sum_{k=0}^{\infty} c_{n,k} z^k.$$

Suppose that $c_{n,k} \to 0$, as $n \to \infty$ for every $k \geq 0$. Can you conclude that $f_n \to 0$ in $H(D)$? If you answer "yes", give a proof. If you answer "no" give an example.