Applied Mathematics Qualifying Written Exam (January 2020)

1 Part I: do 3 out of 4

1. Consider the dynamical system $\dot{x} = rx - x^3$.
   (a) Classify the fix point(s) depending on the choice of r.
   (b) Sketch and explain the bifurcation diagram.

2. Assume a ball carries potential energy $P = mgu(t)$ and is dropped at height $u(t=0)$, attracted by earth's gravity, giving the ball its kinetic energy $K = \frac{1}{2}m\left(\frac{du}{dt}\right)$. Here m denotes the mass of the ball, and g the gravitational constant.
   Use calculus of variations to set up an appropriate functional to derive Newton's law of action. Show all steps in deriving Newton's law.

3. Consider the equation
   $$-\Delta u = 0$$
   on a one-dimensional domain $\Omega = [0, 1]$ with boundary condition $u = 0$ on the left end and $u = 1$ on the right end of $\partial\Omega$.
   (a) Derive the weak form of $-\Delta u = 0$.
   (b) Discretize the domain by introducing three interior vertices equidistantly positioned at distance h and define a basis $\{\Psi_1, \Psi_2, \Psi_3\}$ with piecewise linear basis functions Ψ_i and compact support of width $2h$.
   (c) Derive the linear system by approximating u as a linear combination of Ψ_i (finite element method).

4. Assume a particle moves randomly on a 2D rectangular lattice, where it can jump up/down/left/right with probability p. Let $(x, y) = j\Delta(x, y)$ and $t = k\Delta t$. Let $a(x, y, t + \Delta t)$ be the number of particles at point (x, y) and time $t + \Delta t$. Derive the equation for $a(x, y, t + \Delta t)$ and show that this corresponds to the diffusion equation in the limit case.

2 Part II: do 2 out of 3

1. Newton's law of gravitation states $F = G\frac{M \cdot m}{r^2}$, with F the attractive force between mass M and m, r the distance between the two objects and G the gravitational constant.
   (a) A planet year T is the period of one planet revolving around a star. Use dimensional analysis to determine how T scales with distance R between planet and star.
   (b) Using this result, derive Kepler's law $\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$.
2. Using the theory of curvilinear coordinates, derive a formula for the gradient ∇f of $f : \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates.
3. A rocket car on a straight track is controlled by a rocket on both ends of the car. Show that the optimal control for bringing the rocket car to rest in minimal time requires at most one switching event. To show this, you may use the *Potryagin maximum principle.*