Applied Mathematics Qualifying Written Exam

PART I. Do three of the following four problems.

1. A spring oscillating in a viscous medium has the equation of motion

$$m\frac{d^2y}{dt^2} = -\mu\left(\frac{dy}{dt}\right)^2 - ky$$

with $m, k, \mu > 0$ and with initial conditions $y(0) = A$, $\dot{y}(0) = 0$. Suppose $\mu \ll m$. Scale to obtain a non-dimensional version and identify a small parameter ϵ.

2. Consider the equation

$$\ddot{\theta} + \sin\theta = 0.$$

Determine the stability of the solution $\theta(t) = 0$ to linear perturbation.

3. Determine the Euler-Lagrange equation for a functional of the form

$$J[y] = \int_0^1 F(x, y, y', y'')\, dx.$$

4. Find the Green's function for the problem

$$u'' - u = f(x). \quad 0 < x < 1,$$

with boundary conditions $u(0) = u(1) = 0$.

PART II. Do two of the following three problems.

1. Obtain the first two terms of asymptotic in ϵ approximations to all three solutions of the equation
$$\epsilon x^3 - 3x + 1 = 0.$$

2. Consider the system
$$\begin{aligned} \dot{x} &= y + \mu x, \\ \dot{y} &= x - x^2, \end{aligned}$$
with parameter μ. Find all bifurcations (in μ) and sketch a bifurcation diagram (in μ).

3. Consider the equation
$$u_{xx} + u_{yy} = 0$$
in the upper half plane $y > 0$, and

   (a) the Dirichlet data $u(x,0) = f(x)$,

   (b) the Neumann data $u_y(x,0) = g(x)$,

   where f and g are 2π periodic in x. Assume u is bounded at infinity and also is 2π periodic in x. Find the Fourier transform
$$L\hat{f}(\mathbf{k}) = \hat{g}(\mathbf{k})$$
of the Dirichlet-to-Neumann map, where $\hat{f}$, $\hat{g}$ are the Fourier transforms of f, g, respectively.