Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group (not necessarily finite).

(a) If $H, K \leq G$ are two subgroups of finite index in G, prove that $|G : H \cap K|$ is finite as well.

(b) Let $H, K \leq G$ be arbitrary subgroups. Write $H \sim K$ if and only if $H \cap K$ has finite index in both H and K. Show that $\sim$ defines an equivalence relation on the set of all subgroups of G.

(c) Given a subgroup $H \leq G$, show that $\{x \in G \mid xHx^{-1} \sim H\}$ is a subgroup of G.

I.2 Let F be an algebraically closed field, let n be a positive integer, and let A be an $n \times n$ matrix with entries in F such that $A^3 = A$.

(a) Prove that if F has characteristic zero then A is diagonalizable.

(b) Give an example showing that the conclusion of (a) need not hold in positive characteristic.

I.3 Prove that the intersection of all of the maximal ideals of $\mathbb{Z}[x]$ is the zero ideal.

I.4 Let p be a prime, and let a be a nonzero element of $\mathbb{F}_p$. Prove that $x^p - x + a$ is an irreducible separable polynomial of $\mathbb{F}_p[x]$.

Part II. Do two of these problems.

II.1 Let G be a group of order p^3q, where p and q are primes (not necessarily distinct). Prove that G is not simple. (Do not use Burnside's p^aq^b Theorem.)

II.2 Let F/K be a field extension and let $R = \{f(x) \in F[x] \mid f(0) \in K\}$, the subring of the polynomial ring $F[x]$ consisting of all polynomials with constant term $\in K$. (You need not show that this is a subring of $F[x]$.)

(a) Show that $I = \{f(x) \in F[x] \mid f(0) = 0\}$ is a maximal ideal of R.

(b) Show that the ideal I is finitely generated if and only if the extension F/K has finite degree.

(c) Show that $I/(x)$ is the only prime ideal of $R/(x)$.

II.3 Let F be the splitting field of $x^3 - 2$ over $\mathbb{Q}$, and let G denote the Galois group of F over $\mathbb{Q}$.

(a) Prove that G is isomorphic to the symmetric group S_3.

(b) Give a set of generators of G described as explicit automorphisms of F.