Comprehensive Examination in Algebra
Department of Mathematics, Temple University

January 2008

PART I: Do three of the following problems.

1. Given an additive abelian group A and a positive integer m, set $mA = \{ma : a \in A\}$. Now let A be a finitely generated but not finite additive abelian group. Prove that there exists a postive integer n such that nA is a nonzero free abelian group.

2. Let n be a positive integer, and let N be an $n \times n$ complex matrix. Suppose for every $n \times n$ complex matrix A there exists a complex $n \times n$ matrix B such that $AN = NB$. Prove that N is either the zero matrix or is invertible.

3. Let K be a field. Prove that the polynomial ring in two variables $K[x, y]$ is not a principal ideal domain.

4. Let F be a subfield of $\mathbb{C}$. Suppose that $[F : \mathbb{Q}]$ is an odd positive integer and that F is a normal extension of $\mathbb{Q}$. Prove that F is contained in $\mathbb{R}$.

Part II: Do two of the following problems.

1. Let G be a finite group, and let P be a Sylow p-subgroup of G. Let H be a subgroup of G, and let N be a normal subgroup of G.

   (a) Prove that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$.

   (b) Prove that $P \cap N$ is a Sylow p-subgroup of N.

   (c) Prove that PN/N is a Sylow p-subgroup of G/N.

2. Let R be a ring with identity and suppose that R contains a unique maximal left ideal M.

   (a) Prove that $Ma \subseteq M$ for all $a \in R$, and conclude that M is a two-sided ideal of R.

   (b) Prove that M is equal to the set of non-invertible elements of R. (Recall that an element u of R will be invertible if and only if there exists an element v of R such that $uv = vu = 1$.)

   (c) Prove that M is also the unique maximal right ideal of R.

3. Let K be the splitting field over $\mathbb{Q}$, in $\mathbb{C}$, of $x^4 - 2$. Let $G = \text{Gal}(K/\mathbb{Q})$.

   (a) Determine the order of G, and show that G is isomorphic to the group of symmetries of a plane geometric figure.

   (b) Specify the subfields of K. For each subfield F of K, give field generators over $\mathbb{Q}$, and give the degree $[F : \mathbb{Q}]$.