PH.D. COMPREHENSIVE EXAMINATION
ABSTRACT ALGEBRA SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. a) Let G be a group with a subgroup H and a normal subgroup N. Prove that $H \cap N$ is a normal subgroup of H and that $H/(H \cap N) \cong HN/N$.

b) Use the result in part (a) to prove that a subgroup H of a solvable group is solvable. (Note: A group G is solvable if there exists a subnormal series $\{1\} = N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_2 \triangleleft N_1 \triangleleft N_0 \triangleleft G$, where, for each $i = 1, \ldots, k$, N_{i-1}/N_i is abelian.)

I.2. Let R be an Euclidean domain with unity.

a) Prove that R is a principal ideal domain.

For r and s in R, let $\langle r, s \rangle = \{rm + sn \mid m,\ n \in R\}$

b) Prove that $\langle r, s \rangle$ is an ideal of R.

c) Suppose that $R = Q[x]$ (where Q represents the field of rational numbers), and that $r = x^2 - 3x + 2$ and $s = x^3 - 9x^2 + 23x - 15$. Then, by parts (a) and (b), $\langle r, s \rangle = \langle p \rangle$, for some $p \in Q[X]$. Find p. (Show all work and justify your answer.)

I.3. Let F be a field. Then $F[x]$ is a commutative ring with unity. (You may accept this without proof.)

a) Show that $F[x]$ is an integral domain but that it cannot be a field.

Let E be an extension field of F. For $\alpha \in E$, let $\phi : F[x] \to E$ be a homomorphism which fixes the elements of F and which maps x into α.

b) Describe the kernel of ϕ.

c) Prove that if α is algebraic then the image of ϕ is a subfield of E.

d) For $F = Q$ (the rationals) and for $\alpha = \sqrt[3]{2}$, describe the image of ϕ as a subfield of R (the reals). (I.e., find a basis and a unique representation for each element.)

I.4. Let V be a finite dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. Fix a non-zero vector u in V and define a mapping $T : V \to V$ by $T(v) = \langle u, v \rangle u$.

a) Show that T is a linear transformation.

b) Find the characteristic poynomial, eigenvalues, minimal polynomial, and Jordan canonical form of T.

c) If, instead of T, we consider the mapping $F : V \to V$ given by $F(v) = \langle u, v \rangle v$, explain why the analogues of the questions asked in part b) are not meaningful for F.

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Part II. Do two (2) of these problems.

II.1. a) Find all groups of order 325.
b) Find all groups of order 22.

II.2. Let A be an $m \times n$ matrix over a field F.

a) Show that the rank of A is equal to the smallest integer r such that A can be factored as $A = BC$ for suitable matrices B and C of sizes $m \times r$ and $r \times n$ respectively.
b) Use part (a) to deduce the familiar fact that "row rank = column rank".

II.3. Let F be a finite field with q elements.

a) Prove that the product of the non-zero elements of F is -1.
b) Prove that if q is even then every element of F is a square; and that if q is odd, then the set of non-zero squares of F is a subgroup of index 2 of the group of non-zero elements of F.