Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let S_n be the symmetric group on n letters and G be an abelian subgroup of S_n that acts transitively on $\{1, 2, \ldots, n\}$.

a) Prove that the order of G is n.

b) Give an example of an abelian subgroup $G \leq S_n$ for some n such that G acts transitively on $\{1, 2, \ldots, n\}$ and is not cyclic. (Please justify these two properties.)

I.2 Suppose that R is a commutative ring such that for every $x \in R$, there is some natural number $n > 1$ such that $x^n = x$.

a) Prove that every prime ideal of R is maximal.

b) Give an example of a commutative ring R with the above property and such that R is non-trivial and R is not a field.

I.3 Let V be a finite-dimensional vector space over the field $\mathbb{R}$ with $\dim_{\mathbb{R}} V \geq 3$ and let $T\colon V \to V$ be a linear operator. Show that there exists a subspace $W \subseteq V$ with $\{\vec{0}\} \neq W \subsetneqq V$ such that $T(W) \subseteq W$.

I.4 Let F/K be a field extension and let $R := K + xF[x]$ be the set of all polynomials in the polynomial ring $F[x]$ such that the constant term belongs to K. Put $I := xF[x]$.

(a) Show that R is a subring of $F[x]$ and I is an ideal of R.

(b) Show that I is a finitely generated ideal of R if and only if the field extension F/K is finite.

Part II. Do two of these problems.

II.1 Let G be a group of order p^3, where p is prime. Determine all possibilities for the number of conjugacy classes in G and their sizes.

II.2 Let R be a commutative ring and let M be a maximal ideal of R.

(a) Assuming M to be principal, show that there is no ideal I of R such that $M^2 \subsetneqq I \subsetneqq M$.

(b) Give an example to show that (a) is false if M is not assumed to be principal.

II.3 Consider the subfields $F = \mathbb{Q}(\sqrt[8]{2}, i)$ and $K = \mathbb{Q}(\sqrt{2})$ of $\mathbb{C}$, with $i := \sqrt{-1}$ as usual. Show that F/K is Galois and prove that $\mathrm{Gal}(F/K)$ is isomorphic to the dihedral group D_4 of order 8.