Comprehensive Examination in Algebra
Department of Mathematics, Temple University

August 2020

Part I. Do three of these problems.

I.1 Let p be a prime.

a) Prove that every group of order p^2 is abelian.

b) Construct a non-abelian group of order p^3.

c) Let G be a non-abelian group of order p^3 and let $\mathcal{Z}(G)$ denote its center. Show that $G/\mathcal{Z}(G) \cong \mathcal{Z}_p \oplus \mathcal{Z}_p$, where $\mathcal{Z}_p$ is the cyclic group of order p.

I.2 Let N, K be integers ≥ 2 and assume that N divides K. Consider the canonical ring epimorphism $\varphi : \mathbb{Z}/K\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$, defined by $\varphi(q + K\mathbb{Z}) = q + N\mathbb{Z}$. Furthermore, let $(\mathbb{Z}/K\mathbb{Z})^\times$ and $(\mathbb{Z}/N\mathbb{Z})^\times$ denote the group of units of the rings $\mathbb{Z}/K\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$, respectively. Prove that the group homomorphism

$$\varphi\big|_{(\mathbb{Z}/K\mathbb{Z})^\times} : (\mathbb{Z}/K\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$$

is onto.

I.3 Let R be a (commutative) integral domain and $r \in R$ be a non-zero, non-unit, irreducible element.

a) If R is a UFD, is the quotient ring $R/(r)$ also a UFD?

b) If R is a PID, is the quotient ring $R/(r)$ also a PID?

Please justify your answers.

I.4 Let F be a field and $A \in \mathrm{Mat}_n(F)$, i.e. A in an $n \times n$-matrix with entries in F. Furthermore, let $a_1 \,|\, a_2 \,|\, \ldots \,|\, a_m$ be the invariant factors of A.

a) Let $f \in F[x]$ and prove that $f(A) = 0 \iff$ the polynomial a_m divides f.

b) Give an example of $A \in \mathrm{Mat}_n(F)$ (with $n \geq 2$) whose characteristic polynomial coincides with its minimal polynomial a_m.

c) Give an example of $A \in \mathrm{Mat}_n(F)$ (with $n \geq 2$) whose characteristic polynomial does not coincide with its minimal polynomial a_m.

Please justify your statements.

Part II. Do two of these problems.

II.1 Let G be a group of order $595 = 5 \cdot 7 \cdot 17$ and let n_p denote the number of Sylow p-subgroups of G.

a) Show that $n_5 = 1$ and that at least one of n_7 or n_{17} must be 1.

b) Show that there is a normal subgroup $N \lhd G$ with $|N| = 7 \cdot 17$.

c) Show that N is cyclic: every group of order $7 \cdot 17$ is cyclic.

d) Conclude that G is cyclic.

II.2 Let R be a (commutative) integral domain and I a non-zero ideal of R.

a) Prove that I is a free R-module $\iff$ I is a principal ideal of R. Conclude that all ideals of R are free R-modules if and only if R is a PID.

b) Give an explicit example of an integral domain R, a free R-module M and a non-zero submodule $N \subset M$ that is not free.

Please justify your statements.

II.3 Let $K \supset F$ be a finite Galois extension with Galois group $G = \mathrm{Gal}(K/F)$. Consider a subgroup $H \leq G$ and the corresponding intermediate field, $E = K^H$. Let $\mathrm{Emb}(E/F)$ denote the set of all F-embeddings from E into $\overline{K}$, the algebraic closure of K.

a) Prove that $\tau(E) \subset K$ for every $\tau \in \mathrm{Emb}(E/F)$.

b) Prove that the assignment

$$\sigma H \mapsto \sigma\big|_E$$

is a bijection from the set G/H of left cosets of H in G to the set $\mathrm{Emb}(E/F)$.