Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group (not necessarily finite) and assume that G has a finite normal subgroup N. Let $C = \{g \in G \mid gn = ng \ \forall\, n \in N\}$ denote the centralizer of N in G. Show that C is a normal subgroup of G and that G/C is finite.

I.2 Consider the subring of $\mathbb{Q}$,

$$R := \mathbb{Z}\Big[\frac{1}{2}\Big] = \Big\{ \frac{z}{2^n} \;\Big|\; z \in \mathbb{Z},\ n \geq 0 \Big\}.$$

Prove that R is a principal ideal domain.

I.3 Let V be a vector space over a field F and let $b\colon V \times V \to F$ be a bilinear form, not necessarily symmetric. Assume that $\dim_F V = n < \infty$ and let B denote the $n \times n$-matrix $\big(b(e_i, e_j)\big)_{i,j}$, where $e_1, e_2, \ldots, e_n$ is a fixed F-basis of V. Show that the subspaces

$$V_1 := \{v \in V \mid b(v', v) = 0 \ \forall\, v' \in V\} \qquad \text{and} \qquad V_2 := \{v \in V \mid b(v, v') = 0 \ \forall\, v' \in V\}$$

of V both have dimension equal to $n - \operatorname{rank} B$.

I.4 Let F be a (finite) Galois extension of $\mathbb{Q}$, and let K be a (finite) Galois extension of F. Must K be a Galois extension of $\mathbb{Q}$? Justify your answer with a proof or a counter example.

Part II. Do two of these problems.

II.1 Let G be a finite group. We let $\mathscr{Z}(G)$ denote the center of G and, for any subgroup $H \leq G$, we let $\mathbf{C}_G(H) = \{g \in G \mid gh = hg \ \forall\, h \in H\}$ denote the centralizer of H. Prove:

(a) If the prime p does not divide the order of $G/\mathscr{Z}(G)$, then p does not divide the size of any conjugacy class of G.

(b) If p does not divide the size of the conjugacy class $\mathcal{C} \subseteq G$, then $\mathcal{C} \cap \mathbf{C}_G(P) \neq \varnothing$ for any Sylow p-subgroup $P \leq G$.

(c) Conclude from (b) that the converse of (a) holds: If p does not divide the size of any conjugacy class of G, then p does not divide the order of $G/\mathscr{Z}(G)$.

Hint: For (c), you may use the following standard fact without proof: If $H \lneqq G$ is a proper subgroup of G, then the union of the conjugates gHg^{-1} $(g \in G)$ is a proper subset of G.

II.2 Let F be a field, and let V be a vector space with (countably infinite) basis $\{v_1, v_2, v_3, \ldots\}$. Let R denote the ring $\mathrm{End}_F(V)$ of F-linear transformations from V to itself.

(a) Define $x, y \in R$ by $x(v_1) = 0$, $x(v_i) = v_{i-1}$ $(i > 1)$ and $y(v_i) = v_{i+1}$ $(i \geq 1)$. Show that $xy = 1$, the multiplicative identity for R, but that $yx \neq 1$ in R.

(b) Recall that an element e in a ring is called an *idempotent* if $e^2 = e$. Now put $e_i := y^i x^i$ and $f_i = e_i - e_{i+1}$ $(i \geq 0)$. Show that all e_i and f_i are nonzero idempotents of R and that $f_i f_j = 0$ when $i \neq j$. Conclude that $\bigoplus_{i=0}^{\infty} Rf_i$ is an infinite direct sum of nonzero left ideals of R.

II.3 Let n be an integer ≥ 3 and $\zeta := e^{\frac{2\pi i}{n}}$. Prove that $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$ and that the Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$ is isomorphic to $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}$. Use this result to compute the minimal polynomial of $\zeta_8 := e^{\frac{2\pi i}{8}}$ over $\mathbb{Q}$.