

**PhD Algebra Exam**  
**Fall 1994**

Part I: Do three of these problems.

1. Find all the integers which are orders of elements of the alternating group  $A_5$ . Count how many elements of  $A_5$  are of each of those orders and describe all the elements of each order.

2. Give short explanations:

- (1) Why is a field necessarily a p. i. d.?
- (2) Why is every subgroup of a solvable group solvable?
- (3) Why is a field extension of finite degree necessarily algebraic?
- (4) Why is the number of elements of a finite field necessarily a prime power?
- (5) Why is an abelian simple group necessarily of prime order?

3. Suppose that  $V$  is a finite dimensional vector space,  $T$  a nilpotent linear transformation on  $V$ . Let  $n = \dim V$ . Show that  $\dim T^k(V) \leq n - k$ , and thus  $T^n = 0$ . What does it say about the Jordan normal form of  $T$  if all the inequalities are equality? Give a non-trivial example of a  $T$  for which some of the inequalities are not equality.

4. Consider the rings  $A = \mathbb{Q}[x]/(x^2 - 2x)$ ,  $B = \mathbb{Q}[x]/(x^2 - 1)$ ,  $C = \mathbb{Q}[x]/(x^2)$ . ( $\mathbb{Q}$  = rational numbers.) Show that  $A$  and  $B$  are isomorphic, but  $B$  and  $C$  are not isomorphic.

Part II: Do two of these problems.

5. Let  $F$  be the finite field with  $p$  elements ( $p$  is a prime), and let  $GL(n, F)$  be the group of invertible  $n \times n$  matrices with entries in  $F$ .

- (1) Determine the order of  $GL(2, F)$ .
- (2) Find a  $p$ -Sylow subgroup of  $GL(2, F)$ .
- (3) Determine the order of  $GL(3, F)$ .

6. Show that the only group of order 8 which is isomorphic to a subgroup of the symmetric group  $S_4$  is the dihedral group  $D_4$ .

7. Determine the structure of  $\mathbb{Z}_{15}^\times$ , the group of units of  $\mathbb{Z}_{15}$  (= integers mod 15). Let  $K$  be the cyclotomic field  $\mathbb{Q}(z)$ , where  $z$  is a primitive  $15^{\text{th}}$  root of unity. Exhibit an isomorphism of the Galois group  $G(K/\mathbb{Q})$  with  $\mathbb{Z}_{15}^\times$ .