Toward a functorial quantum spectrum for noncommutative algebras

Manuel L. Reyes

Bowdoin College, Department of Mathematics

Algebra Extravaganza! Temple University
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1. Do noncommutative rings have a spectrum?
2. Topology without points?
3. Back to the drawing board: noncommutative discrete objects
4. Toward a quantum spectrum for noncommutative algebras

The spectrum

As usual, a *spectrum* is an assignment

$$\{\text{commutative algebras}\} \to \{\text{spaces}\}$$

Several examples:

- Commutative rings: $\mathsf{Spec}(R) = \{\text{prime ideals of } R\}$
- Commutative C*-algebras: $\mathsf{Spec}(A) = \{\text{max. ideals of } A\}$
- Boolean algebra: $\mathsf{Spec}(B) = \{\text{ultrafilters of } B\} = \mathsf{Hom}(B, \{0, 1\})$

Each instance of Spec is a (contravariant) functor.

This is usually cited in the motivation for noncommutative geometry of various flavors. . .

A noncommutative spectrum?

Question: What is the "noncommutative space" corresponding to a noncommutative algebra?

Why do I care?

- A solution would yield a rich invariant for noncommutative rings.
- Help us "see" which rings are "geometrically nice" (e.g., smooth).
- Quantum modeling: what is the "phase space" of a quantum system?

To make this a rigorous problem, we should first set some ground rules:

(A) Keep the classical construction if the ring is commutative.
(Let's not tell "commutative" geometers how to do their own job!)

(B) Make it a *functorial* construction.
(To ensure it's truly geometric, and to aid computation.)

Can we begin with a set of points?

Naive idea: Maybe we should assign to each ring a topological space and a sheaf of *noncommutative* rings. But this first requires a nonempty underlying set. . .

Challenge: Pick any noncommuative notion of "prime ideal." I bet your spectrum is either (i) not functorial or (ii) empty for some $R \neq 0$.

Were you just unlucky? Could this be fixed by choosing a different spectrum? No!

Theorem (R., 2012): Any functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to the full subcategory $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec must assign the empty set to $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. (Same holds for C*-algebras.)

Proving the obstruction

Theorem: Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec has $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

Why? Suppose $F(R) \neq \varnothing$ for some R, so there exists $\mathfrak{p}_0 \in F(R)$.

Commutative subrings $C \subseteq D \subseteq R$ yield "compatible" primes:

$$F(R) \to F(D) \to F(C)$$
$$\mathfrak{p}_0 \mapsto \mathfrak{p}_D \mapsto \mathfrak{p}_C = \mathfrak{p}_D \cap C$$

So $\mathfrak{p}_0$ yields a subset $\mathfrak{p} = \bigcup \mathfrak{p}_C \subseteq R$ such that, for each commutative subring $C \subseteq R$, we have $\mathfrak{p} \cap C \in \mathrm{Spec}(C)$.

Def: A subset $\mathfrak{p}$ as above is a prime partial ideal of R, and the set of all prime partial ideals of R is $p\text{-}\mathrm{Spec}(R)$. (Note: $p\text{-}\mathrm{Spec}$ is a functor.)

Colorings from prime partial ideals

Thus: Every F extending Spec maps $F(R) = \varnothing \iff p\text{-}\mathrm{Spec}(R) = \varnothing$

New goal: $p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

What if there were some $\mathfrak{p} \in p\text{-}\mathrm{Spec}(\mathbb{M}_3(\mathbb{C}))$?

Lemma: If $q_1 + q_2 + q_3 = I$ is a sum of orthogonal projections in $\mathbb{M}_3(\mathbb{C})$, then two q_i lie in $\mathfrak{p}$, exactly one lies outside.

Observation: Any prime partial ideal induces a "010-coloring" on the projections $\mathrm{Proj}(\mathbb{M}_3(\mathbb{C}))$ (those in $\mathfrak{p}$ are "0" and those outside are "1").

A surprise: This type of coloring has been studied in quantum physics!

The physical motivation was to obstruct certain "hidden-variable theories" of Quantum Mechanics, under the assumption of "non-contextuality."

The Kochen-Specker Theorem

Q: (Roughly) Can all observables be simultaneously given definite values, which are independent of the device used to measure them?

- Observables: self-adjoint matrices $\mathbb{M}_n(\mathbb{C})_{sa}$
- Definite values: function $\mathbb{M}_n(\mathbb{C})_{sa} \to \mathbb{R}$
- "Yes-No" observable: projection $p = p^2 = p^* \in \mathbb{M}_n(\mathbb{C})$, values $\{0, 1\}$

Def: A function $f\colon \operatorname{Proj}(\mathbb{M}_n(\mathbb{C})) \to \{0, 1\}$ is a Kochen-Specker coloring if, whenever $p_1 + \cdots + p_n = I_n$, we have $f(p_i) = 0$ for all but one i.

Equivalently: f is "Boolean whenever there is no uncertainty":

1. $f(0) = 0$ and $f(1) = 1$;
2. $f(p \wedge q) = f(p) \wedge f(q)$ and $f(p \vee q) = f(p) \vee f(q)$ if p and q are "commeasurable" (commute), with $\wedge$ and $\vee$ defined by ranges.

The Kochen-Specker Theorem

Q: (Roughly) "Can all observables be simultaneously given definite values, independent of the device used to measure them?" No!

Kochen-Specker Theorem (1967)

There is no Kochen-Specker coloring of $\text{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$.

(Proof used clever vector geometry to find a *finite* uncolorable set.)

Corollary: For $n \geq 3$, $p\text{-Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

And as outlined above, this directly proves:

Theorem: Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to $\mathbf{cRing}^{\mathrm{op}}$ is isomorphic to Spec has $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

Kochen-Specker theorem for integer matrices

What is so special about $\mathbb{C}$? How about other fields? Or universally:

Q: For F as above, must $F(\mathbb{M}_n(\mathbb{Z})) = \varnothing$ for $n \geq 3$?

As before, reduce to the "universal" functor $F = p\text{-}\mathrm{Spec}$, and a prime partial ideal induces a Kochen-Specker coloring of idempotent matrices.

Theorem (Ben-Zvi, Ma, R.; with special thanks to Chirvasitu)

There is no Kochen-Specker coloring of $\mathrm{Idpt}(\mathbb{M}_n(\mathbb{Z}))$ *for any* $n \geq 3$.

Corollary: Let R be any ring, and let $n \geq 3$.

- There is no KS coloring of the idempotents of $\mathbb{M}_n(R)$.
- $p\text{-}\mathrm{Spec}(\mathbb{M}_n(R)) = \varnothing$.
- For any F extending Spec, we also have $F(\mathbb{M}_n(R)) = \varnothing$.

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Topology without sets

We can't find a spectrum built out of points. But there are "point-free" ways to do topology!

$$\begin{array}{ccc} \text{topological spaces} & \longrightarrow & \text{sets} \\ \downarrow & & \\ \text{“pointless” spaces} & & \\ \downarrow & & \\ \text{categories of sheaves} & & \end{array}$$

Perhaps points are the real problem, so that one of these more exotic approaches could bypass the obstruction?

Avoiding the obstruction with pointless topology?

Pointless topology treats spaces and sheaves purely in terms of their lattices of open subsets, called locales, forming a category **Loc**

Can we avoid the obstruction by "throwing away points?" No!

Theorem (van den Berg & Heunen, 2012)

Any functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *whose restriction to* $\mathbf{cRing}^{\mathrm{op}}$ *is isomorphic to* Spec *(considered as a locale) must assign the trivial locale to* $\mathbb{M}_n(R)$ *for any ring* R *with* $\mathbb{C} \subseteq R$ *and any* $n \geq 3$*. (The same holds for C*-algebras.)*

Cor: [Ben-Zvi, Ma, R.] This obstruction still holds with any ring R.

On to the next idea: The space $\mathrm{Spec}(R)$ has a *sheaf of rings* $\mathcal{O}_{\mathrm{Spec}(R)}$, whose ring of global sections is isomorphic to R. And we can axiomatize sheaves without a space!

Interpreting sheaves of rings in $\mathbf{Sh}(X)$

$\mathbf{Sh}(X)$ as a category:

- Has all finite products.
- Thus, has a terminal object 1_X (single point at each open).
- Global sections: $\Gamma(X, \mathcal{O}) = \mathrm{Hom}(1_X, \mathcal{O})$.

Sheaf of rings $\mathcal{O}$ = "ring object" $\mathcal{O}$ in $\mathbf{Sh}(X)$: zero, unity, addition, multiplication, and negatives can be phrased as *morphisms* in $\mathbf{Sh}(X)$:

$$0, 1 \in \mathrm{Hom}(1_X, \mathcal{O}), \quad +, \times \in \mathrm{Hom}(\mathcal{O} \times \mathcal{O}, \mathcal{O}), \quad \text{and} \quad - \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}),$$

and the axioms of an associative ring can be written as commutative diagrams in $\mathbf{Sh}(X)$.

Generalize: Replace $\mathbf{Sh}(X)$ by a category $\mathcal{X}$ with finite products.

Global sections: For $A \in \mathcal{X}$, we *define* $\Gamma(\mathcal{X}, A) := \mathrm{Hom}_{\mathcal{X}}(1_{\mathcal{X}}, A)$.

Ringed categories

The following generalizes ringed spaces/locales/toposes.

Def: A ringed category is a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ where $\mathcal{X}$ is a category with finite products and $\mathcal{O}_\mathcal{X}$ is a ring object in $\mathcal{X}$.
A morphism of ringed categories $f\colon (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a pair:

1. functor $f_*\colon \mathcal{X} \to \mathcal{Y}$ preserving finite products and global sections,
2. morphism $\underline{f}\colon \mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{X}$ of ring objects in $\mathcal{Y}$.

We get a global sections functor:

$$\begin{aligned} \Gamma\colon \mathbf{RingedCat}^{\mathrm{op}} &\to \mathbf{Ring} \\ (\mathcal{X}, \mathcal{O}_\mathcal{X}) &\mapsto \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}). \end{aligned}$$

The goal would be to replicate $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \cong R$.

Another obstruction

Theorem [R. 2014]: Let $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{RingedCat}$ be a functor whose restriction to commutative rings is isomorphic to Spec. Suppose that there are natural homomorphisms

$$R \to \Gamma(F(R))$$

for each ring R, which restrict to the canonical isomorphisms $R \cong \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for commutative R.
Then for any ring k and integer $n \geq 2$, the ringed category $F(\mathbb{M}_n(k))$ has

$$\Gamma(F(\mathbb{M}_n(k))) = 0.$$

There is also a similar obstruction if we try to view noncommutative rings as sheaves on a noncommutative extension of the big Zariski site.

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In search of "noncommutative sets"

The obstructions suggest to me we do not yet understand discrete noncommutative objects.

If we strip a "commutative" space of its geometry, we are left with its underlying set. But if we strip a noncommutative space of its geometry, then what *noncommutative* discrete structure remains?

$$\begin{array}{ccccc}
\{\text{commutative algebras}\} & \xrightarrow{\mathsf{Spec}} & \{\text{spaces}\} & \xrightarrow{U} & \{\text{sets}\} \\
\big\downarrow & & & & \big\downarrow \\
\{\text{noncommutative algebras}\} & & \longrightarrow & & \{???\}
\end{array}$$

What category should fill in blank above?

Noncommutative sets via function algebras

If we are serious about noncommutative geometry, we might expect:

$$\{\text{“noncommutative sets”}\} \leftrightarrow \{\text{suitable noncommutative algebras}\}$$

Indeed, the functor $X \mapsto k^X$ yields a duality between **Set** and certain topological algebras [Iovanov, Mesyan, R., 2016].

Easier for C*-algebras: The algebra of continuous functions on space X embeds in the algebra of discrete functions as $C(X) \subseteq \mathbb{C}^X$.

Q: Does $C(X) \mapsto \mathbb{C}^X$ extend to a functor $F\colon$ **Cstar** $\to$ **Alg**, with natural embeddings $A \to F(A)$, for some suitable category of $*$-algebras **Alg**?

Discretization of C*-algebras

Necessary condition: Applying $F\colon \mathbf{Cstar} \to \mathbf{Alg}$ to every commutative subalgebra $C(X) \cong C \subseteq A$ induces a commuting square

$$
\begin{array}{ccc}
A & \xrightarrow{\quad \phi \quad} & M = F(A) \\
\big\uparrow & & \big\uparrow \phi_C \\
C(X) & \hookrightarrow & \mathbb{C}^X
\end{array}
$$

Such $\phi\colon A \to M$ with factorizations ϕ_C as above is a discretization of A.

Theorems [Heunen & R., 2017]:

- Every C*-algebra embeds into a non-functorial discretization.
- **Alg** above cannot be the category of AW*-algebras.
- There is a functor that discretizes all algebras embedding in $\mathbb{M}_n(C(X))$. (These are the C*-algebras that are PI algebras.)

But the general question remains open. . .

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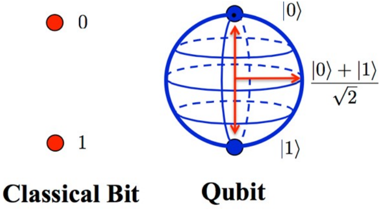
From sets to “quantum sets”

$$\begin{array}{ccc} \{\text{commutative algebras}\} & \xrightarrow{\text{Spec}} & \{\text{sets}\} \\ \big\downarrow & & \big\downarrow \\ \{\text{noncommutative algebras}\} & \longrightarrow & \{???\} \end{array}$$

Taking a cue from quantum mechanics: If X is our set of “states,” we should also allow linear combinations of states: $X \rightsquigarrow kX = \mathrm{Span}(X)$



http://qoqms.phys.strath.ac.uk/research_qc.html

“Quantum sets” for algebras over a field

This vector space $Q = kX$ carries the structure of a coalgebra:

- Comultiplication $\Delta \colon Q \to Q \otimes Q$ given by $x \mapsto x \otimes x$
- Counit $\eta \colon Q \to k$ given by $x \mapsto 1$

Coalgebra maps correspond to set maps: $\mathbf{Set}(X, Y) \cong \mathbf{Coalg}(kX, kY)$.
Gives a full and faithful embedding $\mathbf{Set} \hookrightarrow \mathbf{Coalg}$.

Therefore: We view a coalgebra (Q, Δ, η) as a “quantum set” (over k).
Its *algebra of observables* is the dual algebra $\mathrm{Obs}(Q) = Q^*$.

History: Coalgebras were considered as “discrete objects” by Takeuchi (1974), and in the noncommutative context by Kontsevich-Soibelman (*noncommutative thin schemes*) and Le Bruyn.

Coalgebras in commutative geometry

Every scheme over k has an "underlying coalgebra."

Motivating fact: The underlying set $|X|$ of a Hausdorff space X is the directed limit of its finite discrete subspaces.

Observe: A scheme S finite over k is of the form $S \cong \mathrm{Spec}(B)$ for f.d. algebra B. The functor $S \mapsto \Gamma(S, \mathcal{O}_S)^* \cong B^*$ is an equivalence

$$\{\text{finite schemes over } k\} \xrightarrow{\sim} \{\text{f.d. cocomm. coalg's}\}$$

Def: For a k-scheme X, the coalgebra of distributions is

$$\mathrm{Dist}(X) = \varinjlim \Gamma(S, \mathcal{O}_S)^*,$$

where S ranges over the closed subschemes of X that are finite over k. This gives a functor $\mathrm{Dist}\colon \mathbf{Sch}_k \to \mathbf{Coalg}$.

Local nature of distributions

It's best to restrict to the case where X is (locally) of finite type over k.

Distributions supported at a closed point x of such X have been defined in the literature on algebraic groups:

$$\mathrm{Dist}(X, x) = \varinjlim (\mathcal{O}_{X,x}/\mathfrak{m}_x^n)^*.$$

This is dual to the completion $\mathrm{Obs}(\mathrm{Dist}(X, x)) \cong \widehat{\mathcal{O}}_{X,x}$.

Theorem: Suppose X is of finite type over k, and let X_0 be its set of closed points.

1. There is an isomorphism of coalgebras $\mathrm{Dist}(X) \cong \bigoplus_{x \in X_0} \mathrm{Dist}(X, x)$
2. If $k = \overline{k}$, then $\mathrm{Dist}(X)$ has a subcoalgebra isomorphic to kX_0.

Moral: $\mathrm{Dist}(X)$ linearizes the set of closed points, and includes the formal neighborhood of each point.

Distributions in the affine case

Affine case: $X = \mathrm{Spec}(A)$ with A an affine commutative k-algebra. Distributions given by the Sweedler dual coalgebra

$$\mathrm{Dist}(X) \cong A^\circ := \varinjlim (A/I)^*$$

where I ranges over all ideals of finite codimension.

Thesis: For "nice" affine algebras over k, then the functor $A \mapsto A^\circ$ is a suitable candidate for a quantized maximal spectrum.

"Nice" means many f.d. representations: We say A is fully residually finite if every quotient A/I is residually finite-dimensional.

Examples:

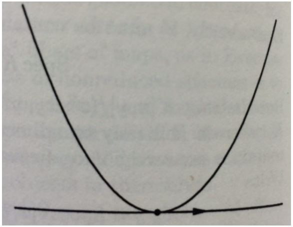
- Affine, noetherian PI algebras
- In particular, lots of "quantum algebras" at roots of unity
- Just infinite algebras (A/I finite-dim'l for all $I \neq 0$)

Glimpses of some quantum spectra

Ex: The ring of dual numbers $A = k[t]/(t^2)$ has $A^\circ = kx \oplus k\varepsilon$ with

- $\Delta(x) = x \otimes x$ and $\Delta(\varepsilon) = x \otimes \varepsilon + \varepsilon \otimes x$
- $\eta(x) = 1$ and $\eta(\varepsilon) = 0$

Here x is like a point and ε is "infinitesimal fuzz."



Eisenbud & Harris, *The Geometry of Schemes*

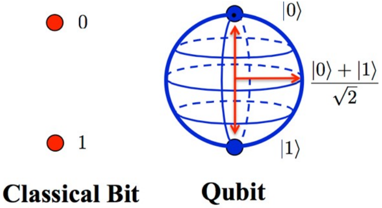
This is closer to the geometers' picture than $\mathrm{Spec}(A) = \mathrm{Max}(A) = \{pt\}$!

Glimpses of some quantum spectra

Ex: The qubit is the matrix coalgebra $\mathbb{M}^2 = (\mathbb{M}_2(k))^\circ = \bigoplus_{i,j=1}^{2} kE^{ij}$

- $\Delta(E^{ij}) = E^{i1} \otimes E^{1j} + E^{i2} \otimes E^{2j}$
- $\eta(E^{ij}) = \delta_{ij}$

There is a "disentangling" morphism from the qubit to the classical bit Dist(Spec(k^2)), roughly sending each E^{ii} to a point and $E^{12}, E^{21} \mapsto 0$.



But we many morphisms to the classical bit, one for every basis of k^2!

Future work

Work currently in progress:

- Describing the underlying coalgebra of a "noncommutative Proj(S)," still assuming that S has "many" f.d. representations.
- Developing strategies to compute A°

Several questions that eventually need to be addressed:

- Doing geometry with coalgebras: how to "topologize" them and define sheaves?
- What is a "quantum scheme of finite type over k" in this context?
- Could this approach extend to algebras that are not residually finite?
- Could it even extend to rings that are not algebras over a field?

Thank you!